An Introduction to Large Random Matrices and Some Applications

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Motivations and aim of this course

Large Random Matrix Theory is an important tool in applied maths:

- High dimensional statistics, modelling
- Spectral graph theory
- Electrical engineering (wireless communication)
- Theoretical ecology
- Mathematical finance
- .. also more theoretical stuff: operator theory (free probability)

Basic reasons

- Matrices are important in applications
- In high dimension, the matrix of interest is often unknown and a RM might be an acceptable educated guess

Aim of these lectures

- RMT is at the crossroads of proba, stats, combinatorics, complex analysis, etc.
- We intend to **lower the entry price** to RMT
- will present classical results, technical means, some applications.

Resources

Figure: Resources (slides, lecture notes, etc.) available here.

Large Random Matrices

Random matrices

It is a $N \times N$ matrix

$$\boldsymbol{Y}_N = \begin{bmatrix} Y_{11} & \cdots & Y_{1N} \\ \vdots & & \vdots \\ Y_{N1} & \cdots & Y_{NN} \end{bmatrix}$$

whose entries $(Y_{ij};\ 1 \leq i, j \leq N)$ are random variables.

Matrix features

Of interest are the following quantities

- $\boldsymbol{Y}_N$'s **spectrum** $(\lambda_i,\, 1 \leq i \leq N)$ and **eigenvectors (= eigenstructure)**
- **Extreme eigenvalues** $\lambda_{\min}$ and $\lambda_{\max}$ if spectrum is real, etc.
- Some information **beyond the eigenstructure of the matrix**.

Asymptotic regime

Often, the description of the previous features **takes a simplified form** as

$$\boxed{N \to \infty}$$

Moreover this regime is **of interest in many applications**.

Large Random Matrices: Wigner Matrices

Matrix model

Let $\boldsymbol{X}_N = (X_{ij})$ a symmetric $N \times N$ matrix with i.i.d. (real) entries **on and above** the diagonal with

$$\mathbb{E}X_{ij} = 0 \text{ and } \mathbb{E}\,|X_{ij}|^2 = 1$$

and $X_{ij} = X_{ji}$ (for symmetry).

- consider the spectrum of **Wigner matrix** $\boxed{\boldsymbol{Y}_N = \frac{\boldsymbol{X}_N}{\sqrt{N}}}$

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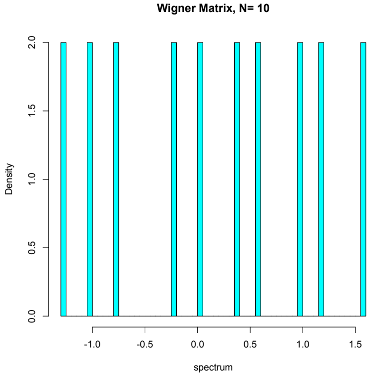

Figure: Histogram of the eigenvalues of $\mathbf{Y}_N$

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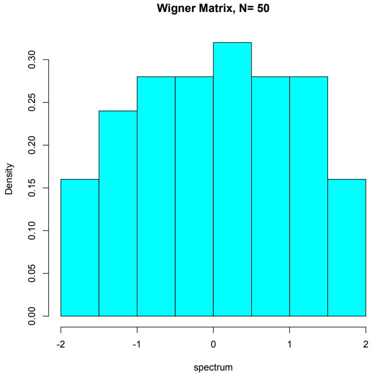

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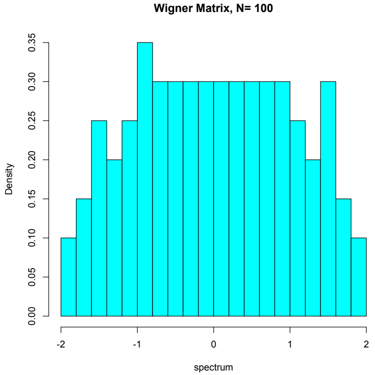

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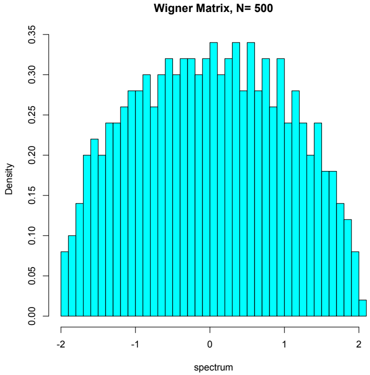

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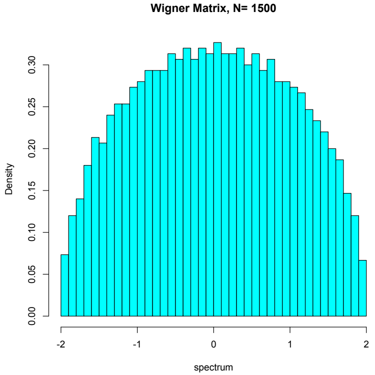

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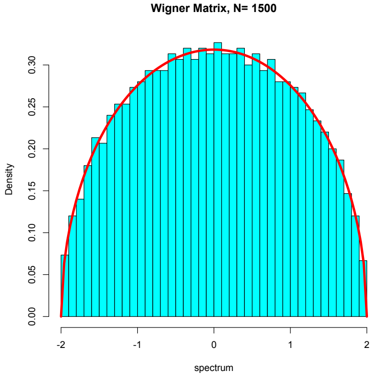

Figure: The semi-circular distribution (in red) with density $x \mapsto \frac{\sqrt{4-x^2}}{2\pi}$

Wigner's theorem (1948)

"The histogram of a Wigner matrix converges to the **semi-circular distribution**"

Matrix model

Let $\boldsymbol{X}_n$ be a $N \times n$ matrix with i.i.d. entries

$$\mathbb{E}X_{ij} = 0 \ , \ \mathbb{E}|X_{ij}|^2 = 1$$

and consider the spectrum of $\frac{1}{n}\boldsymbol{X}_n\boldsymbol{X}_n^*$ in the regime where

$$N, n \to \infty \quad \text{and} \quad \frac{N}{n} \to c \in (0, \infty)$$

dimensions of matrix $\boldsymbol{X}_n$ of the **same order**

Large Covariance Matrices

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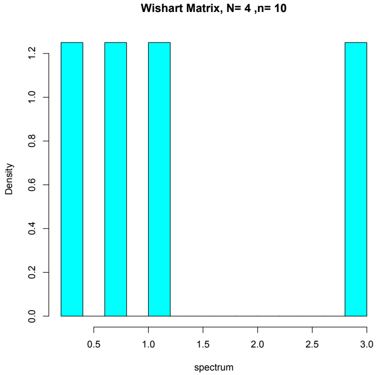

Figure: Spectrum's histogram - $\frac{N}{n} = 0.4$

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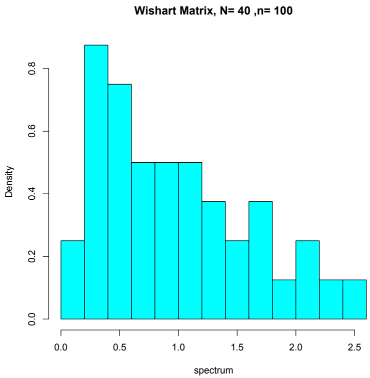

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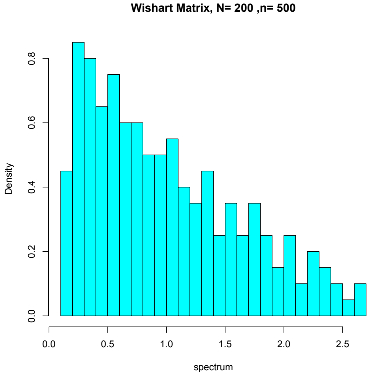

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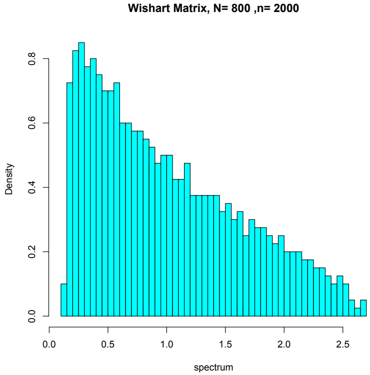

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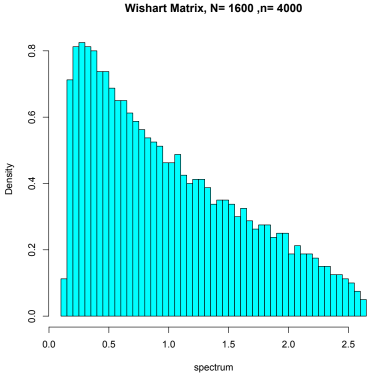

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Large Covariance Matrices : Marčenko-Pastur's theorem

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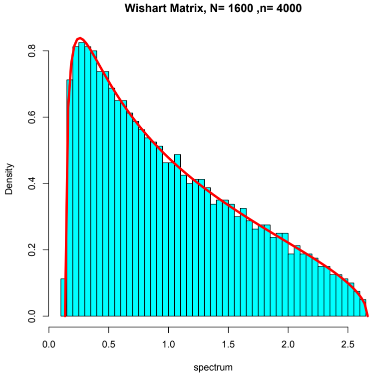

Figure: Marčenko-Pastur's distribution (in red)

Marčenko-Pastur's theorem (1967)

"The histogram of a **Large Covariance Matrix** converges to **Marčenko-Pastur distribution** with given parameter (here **0.4**)"

Matrix model

Let $\boldsymbol{X}_N$ be a $N \times N$ matrix with i.i.d. entries

$$\mathbb{E}X_{ij} = 0 \ , \ \mathbb{E}|X_{ij}|^2 = 1$$

and consider the spectrum of matrix

$$\mathbf{Y}_N = \frac{\boldsymbol{X}_N}{\sqrt{N}}$$

as $N \to \infty$

- We call it a **Ginibre** model
- In this case, the eigenvalues are **complex**!

Large Non-Hermitian Matrices I: the Circular Law

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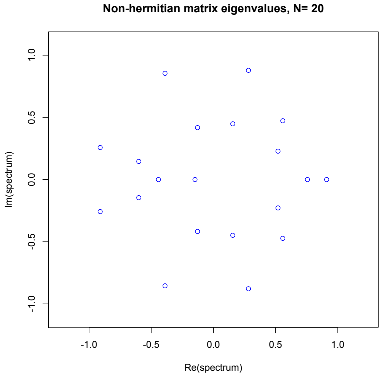

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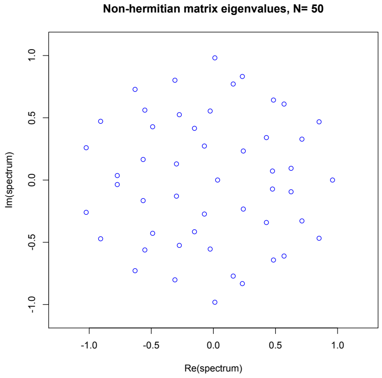

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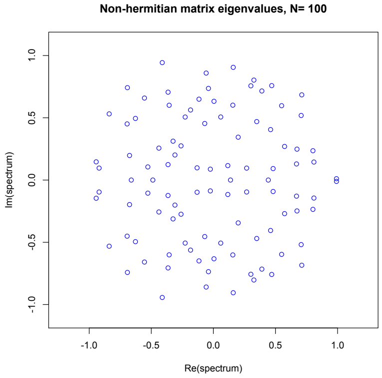

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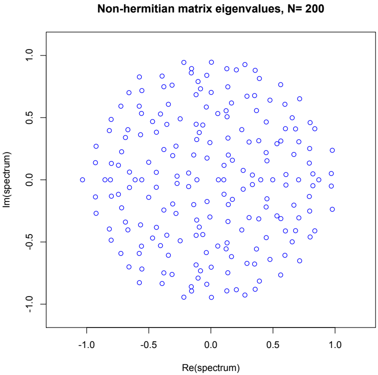

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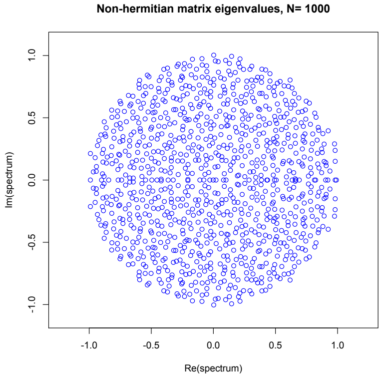

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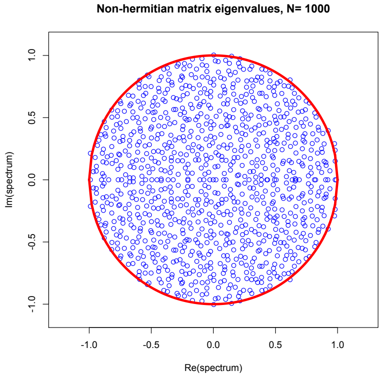

Figure: The circular law (in red)

Theorem: The Circular Law (Ginibre, Metha, Girko, Tao & Vu, etc.)

$\boxed{\text{The spectrum of } \mathbf{Y}_N \text{ converges to } \textbf{the uniform probability on the disc}}$

Large Non-Hermitian Matrices II: the Elliptic Law

Matrix model

Let $\boldsymbol{X}_N$ be a $N \times N$ matrix with standardized entries and:

- the following variables are independent

$$\{X_{ii}, (X_{ij}, X_{ji}),\ i < j, \}$$

- assume the covariance structure

$$\mathrm{cov}(X_{ij}, X_{ji}) = \rho$$

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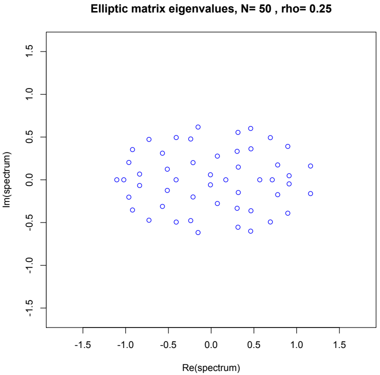

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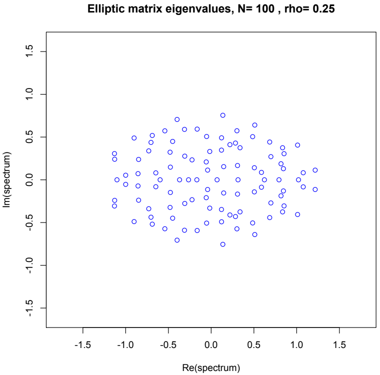

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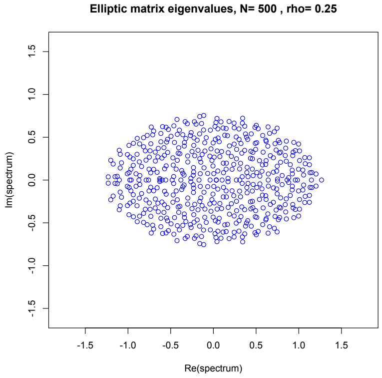

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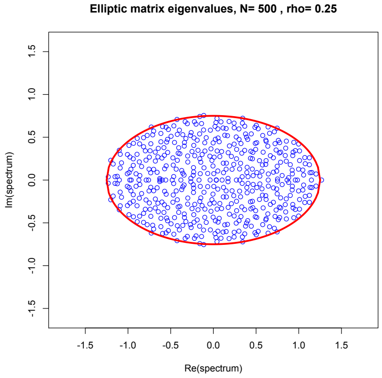

Figure: The elliptic law (in red)

Theorem: The Elliptic Law (Girko, Nguyen & O'Rourke, etc.)

$\boxed{\text{The spectrum of } \mathbf{Y}_N \text{ converges to } \textbf{the uniform probability on the ellipse}}$

Spiked Models

Small perturbations (to be specified) of standard models can modify the behavior of extreme eigenvalues.

- Such models are called **spiked models**,
- Very useful in applications,
- Example: Spikes $=$ signal vs MP spectrum $=$ noise

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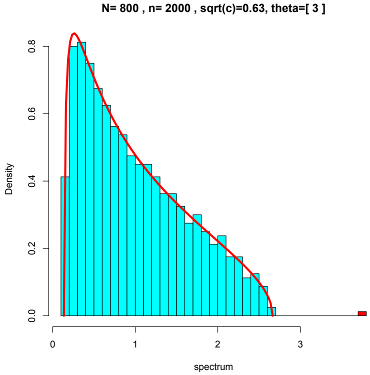

Figure: Perturbated MP with single spike

Large Random Matrices: Spiked Models

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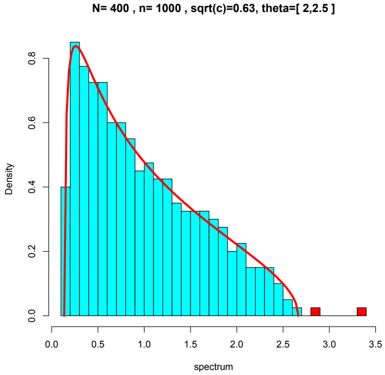

Figure: Perturbated MP with double spike

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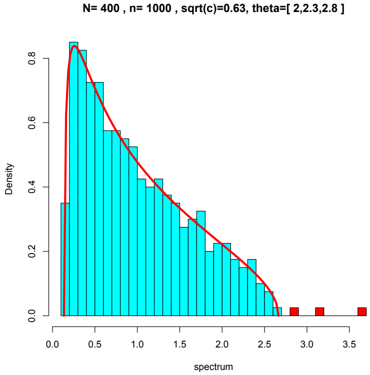

Figure: Perturbated MP with triple spike

A word on the normalization

Consider $\boldsymbol{X}_N = (X_{ij})$ a $N \times N$ symmetric matrix with i.i.d. (real) entries on and above the diagonal,

$$\mathbb{E}X_{ij} = 0 \qquad \text{and} \qquad \mathrm{var}(X_{ij}) = 1\,.$$

Without normalization

$$\frac{1}{N}\sum_{i=1}^{N}\lambda_i^2(\boldsymbol{X}_N) = \frac{1}{N}\mathrm{Trace}\boldsymbol{X}_N^2 = \frac{1}{N}\sum_{i,j=1}^{N}|X_{ij}|^2 \quad \nearrow \quad +\infty \qquad (N\to\infty)$$

With normalization

$$\frac{1}{N}\sum_{i=1}^{N}\lambda_i^2\left(\frac{\boldsymbol{X}_N}{\sqrt{N}}\right) = \frac{1}{N}\mathrm{Trace}\left(\frac{\boldsymbol{X}_N}{\sqrt{N}}\right)^2 = \frac{1}{N^2}\sum_{i,j=1}^{N}|X_{ij}|^2 \quad \longrightarrow \quad 1 \qquad (N\to\infty)$$

Hence the heuristics

$$\boxed{\lambda_i\left(\frac{\boldsymbol{X}_N}{\sqrt{N}}\right) \quad \simeq \quad \mathcal{O}(1)}$$

Eigenstructure

Eigenvectors and eigenvalues

Given a $N \times N$ matrix $\mathbf{A}$ we are interested in its **eigenvalues** λ

$$\boxed{\mathbf{A}\vec{u} = \lambda\vec{u} \ , \quad (\vec{u} \neq 0)}$$

and its associated **eigenvectors** $\vec{u}$.

Remarks

- $\lambda \in \mathbb{C}$ is an eigenvalue of A iff $\boxed{\det(A - \lambda I) = 0}$,
- The relationship between an eigenvalue and the entries of the corresponding matrix is **very involved**,
- We call the **spectrum** of matrix A the set of its eigenvalues counted with their multiplicities (as roots of the polynomial $P(\lambda) = \det(A - \lambda I)$)

Important question

- How can we infer properties on the spectrum of matrix A based on the entries A_{ij} of the matrix? [moment method, Stieltjes transform ..]

The spectral theorem

The spectral theorem - complex case

if $\mathbf{A}$ is hermitian:

$$\mathbf{A} = \mathbf{A}^* \quad \Leftrightarrow \quad [\mathbf{A}]_{ij} = \overline{[\mathbf{A}]}_{ji}$$

then $\mathbf{A}$ is diagonalizable with real eigenvalues:

$$\boxed{\mathbf{A} = \mathbf{U}^* \mathbf{\Lambda} \mathbf{U} \ , \quad \mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}_N}$$

with $\mathbf{U}$ unitary matrix and $\mathbf{\Lambda}$ real diagonal.

The spectral theorem - real case

If $\mathbf{A}$ is symmetric that is $\mathbf{A} = \mathbf{A}^T$, then

$$\boxed{\mathbf{A} = \mathbf{O}^T \mathbf{\Lambda} \mathbf{O} \ , \quad \mathbf{O}\mathbf{O}^T = \mathbf{O}^T\mathbf{O} = \mathbf{I}_N}$$

where $\mathbf{O}$ is (real) **orthogonal**.

Example

- Let $P \in \mathbb{R}[X]$ (=polynomial with real coefficients). Let $\boldsymbol{A}$ hermitian and $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^*$ then

$$[P(\boldsymbol{A})]^* = P(\boldsymbol{A}) \quad \text{and} \quad P(\boldsymbol{A}) = \boldsymbol{U} P(\boldsymbol{\Lambda}) \boldsymbol{U}^* \, .$$

The spectral measure of a matrix $\mathbf{A}$

.. also called the **empirical measure of the eigenvalues**.

- It is a central object to express the limiting properties of the spectrum.

The Dirac measure

We define a **probability measure** δ_x over $\mathbb{R}$ by

$$\delta_x([a,b]) = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{else} \end{cases}$$

The spectral measure

If $\mathbf{A}$ is $N \times N$ hermitian with eigenvalues $\lambda_1, \cdots, \lambda_N$ then its **spectral measure** is:

$$L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i} \qquad \Rightarrow \qquad L_N([a,b]) = \frac{\#\{\lambda_i \in [a,b]\}}{N}$$

Otherwise stated

$\boxed{L_N([a,b]) \text{ is the \textbf{proportion} of eigenvalues of } \mathbf{A} \text{ in } [a,b].}$

Weak convergence of probability measures I

Let $\mu_n (n \geq 1)$ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, μ a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Weak convergence

We say that μ_n weakly converges towards μ iff for every **bounded continuous** function $f : \mathbb{R} \to \mathbb{R}$ we have:

$$\int_{\mathbb{R}} f(x)\mu_n(dx) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} f(x)\mu(dx) \quad \text{Notation:} \quad \boxed{\mu_n \xrightarrow[n\to\infty]{w} \mu}$$

Remarks

- If $\mu_n \xrightarrow[n\to\infty]{w} \mu$ then μ is a probability measure: $\mu(\mathbb{R}) = 1$,
- If $\mu_n = \mathcal{L}(X_n)$ and $\mu = \mathcal{L}(X)$ then: $\mu_n \xrightarrow[n\to\infty]{w} \mu \iff X_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

Weak convergence of probability measures II

Let μ_n be a family of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Tightness

The sequence (μ_n) is **tight** iff for every $\varepsilon > 0$ there exists a compact set K_ε such as

$$\sup_n \mu_n(K_\varepsilon^c) \leq \varepsilon \quad \Longleftrightarrow \quad \inf_n \mu_n(K_\varepsilon) \geq 1 - \varepsilon \,.$$

(basically, up to ε the μ_n have a common support K_ε)

Theorem (weak vs vague convergence)

Let μ a measure. The following statements are equivalent:

- $\mu_n \xrightarrow[n\to\infty]{w} \mu$,
- $\mu_n \xrightarrow[n\to\infty]{v} \mu$ and (μ_n) is tight,
- $\mu_n \xrightarrow[n\to\infty]{v} \mu$ and μ is a probability measure.

Weak convergence - the moment method I

Characterization by moments

Let μ a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and assume that for all

$$k \in \mathbb{N}\,, \quad m_k = \int_{\mathbb{R}} x^k \mu(dx) \in \mathbb{R}\,.$$

We say that μ is **uniquely characterized by its moments** if it is the unique measure with moments given by the m_k's.

Theorem (Carleman)

Probability measure μ is uniquely characterized by its moments iff $\boxed{\sum_{k\geq 1} m_{2k}^{-\frac{1}{2k}} = \infty}$

Theorem (sufficient condition)

Probability measure μ is uniquely characterized by its moments if

$$\limsup_k \left(\frac{m_{2k}}{(2k)!} \right)^{\frac{1}{2k}} < \infty\,.$$

Remarks

- If μ has a bounded support then it is uniquely characterized by its moments.
- If $\mu \sim \mathcal{N}(0,1)$ then it is uniquely characterized by its moments.

Weak convergence - the moment method II

Theorem

Let $\mu_n (n \geq 1)$ and μ probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with all their moments. Assume that

- μ is uniquely determined by its moments,
- the convergence of the moments holds

$$\forall k \geq 1, \qquad \int_{\mathbb{R}} x^k \mu_n(dx) \xrightarrow[n \to \infty]{} \int_{\mathbb{R}} x^k \mu(dx)\,.$$

Then

$$\boxed{\mu_n \xrightarrow[n \to \infty]{w} \mu}$$

Why is the moment method important in RMT?

- Let $\boldsymbol{A}$ be $n \times n$ hermitian and L_n its spectral measure:

$$L_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i} \, .$$

- The k-th moment of L_n writes

$$\boxed{\int x^k L_n(dx) = \frac{1}{n} \sum_i \lambda_i^k}$$

- by the spectral theorem, it is also equal to

$$\boxed{\frac{1}{n} \sum_i \lambda_i^k = \frac{1}{n} \mathrm{Trace}(\boldsymbol{A}^k)}$$

  This provides a "simple" relationship between the eigenvalues of $\boldsymbol{A}$ and its entries as:

$$\frac{1}{n} \mathrm{Trace}(\boldsymbol{A}^k) = \frac{1}{n} \sum_{i_1, \cdots, i_k = 1}^{n} A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_k i_1}$$

- This last equation is at the heart of combinatorial techniques developed in RMT.

Spectrum Analysis: The Stieltjes Transform I

Given a probability measure $\mathbb{P}$, its **Stieltjes transform** is an analytic function on

$$\mathbb{C}^+ = \{z \in \mathbb{C}\,,\ \Im(z) > 0\}$$

defined as

$$\boxed{g(z) = \int_{\mathbb{R}} \frac{\mathbb{P}(d\lambda)}{\lambda - z}}\,, \quad z \in \mathbb{C}^+\ ,$$

Examples

1. Dirac measure:

$$\mathbb{P} = \delta_{\lambda_0} \quad \Rightarrow \quad g(z) = \frac{1}{\lambda_0 - z}$$

2. Spectral measure:

$$\mathbb{P} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i} \quad \Rightarrow \quad g(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_i - z}$$

Spectrum Analysis: The Stieltjes Transform II

It has good properties such as

- inverse formulas

$$
\begin{aligned}
\mathbb{P}[a,b] &= \frac{1}{\pi}\lim_{y\downarrow 0}\Im\int_a^b g(x+\mathbf{i}y)\,dx\ , \quad \text{if } \mathbb{P}\{a\}=\mathbb{P}\{b\}=0 \\
\int f\,d\,\mathbb{P} &= \frac{1}{\pi}\lim_{y\downarrow 0}\Im\int_{\mathbb{R}} f(x)g(x+\mathbf{i}y)\,dx\ ,
\end{aligned}
$$

- criterion for the weak convergence of probability measures: let $g_n = ST(\mu_n)$

$$
\forall z\in\mathbb{C}^+\ ,\quad g_n(z)\xrightarrow[n\to\infty]{} g(z) \quad \text{and} \quad g = ST(\mu) \text{ with } \mu\in\mathcal{P}(\mathbb{R})
$$

  is equivalent to

$$
\mu_n \xrightarrow[n\to\infty]{w} \mu\ .
$$

Relation with the resolvent of Large Random Matrices I

- The resolvent of A is

$$\boxed{\boldsymbol{Q}(z) = (A - zI)^{-1}} \qquad z \notin \mathrm{spectrum}(A)\,.$$

- Named resolvent because it solves the equation:

$$\boxed{A\,\boldsymbol{x} = z\,\boldsymbol{x} + \boldsymbol{b}} \qquad \Leftrightarrow \qquad (A - zI)\boldsymbol{x} = \boldsymbol{b} \qquad \Leftrightarrow \qquad \boldsymbol{x} = \boldsymbol{Q}(z)\boldsymbol{b}$$

- its singularities are exactly **eigenvalues** of $\mathbf{A}$.
- Eigen-decomposition of hermitian matrix A yields eigen-decomposition of $\mathbf{Q}$:

$$A = U^*\Lambda U \quad \Rightarrow \quad \mathbf{Q}(z) = \ U^*(\Lambda - zI)^{-1}U$$

$$A = U^*\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} U \quad \Rightarrow \quad \boldsymbol{Q}(z) = U^*\begin{bmatrix} \frac{1}{\lambda_1 - z} & & \\ & \ddots & \\ & & \frac{1}{\lambda_N - z} \end{bmatrix} U$$

Relation with the resolvent of Large Random Matrices

Let A hermitian with eigenvalues (λ_i) and spectral measure $\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$.

Then

$$
\begin{aligned}
g(z) &= \text{Stieltjes transform of } \left(\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}\right) \\
&= \frac{1}{N}\sum_{1}^{N}\frac{1}{\lambda_i - z} \\
&= \frac{1}{N}\mathrm{Trace}\begin{bmatrix} \frac{1}{\lambda_1 - z} & & \\ & \ddots & \\ & & \frac{1}{\lambda_N - z}\end{bmatrix} = \frac{1}{N}\mathrm{Trace}\,(\mathbf{A} - z\boldsymbol{I})^{-1}
\end{aligned}
$$

- $\boxed{\text{The Stieltjes transfom } g \text{ is the } \textbf{normalized trace} \text{ of the resolvent } (A - zI)^{-1}}$
- This represents a simple relationship between the ST of the spectral measure and matrix A. It is the starting point of many techniques to analyze spectral measures.

Wigner's theorem

The semi-circular distribution

Let $\sigma > 0$. The **semi-circular** distribution is given by

$$\boxed{\mathbb{P}_{\mathrm{sc},\sigma^2}(dx) = \frac{1}{2\pi\sigma^2}\sqrt{(4\sigma^2 - x^2)_+}\, dx}$$

Wigner's theorem (sharp assumptions)

- Let X_{ij} $(1 \le i < j \le N)$ i.i.d. centered, $\xrightarrow{X_{ij}} \mathbb{C}$ with $\mathbb{E}|X_{ij}|^2 = \sigma^2 < \infty$.
- Let X_{ii} $(1 \le i \le N)$ i.i.d. centered, $\xrightarrow{X_{ii}} \mathbb{R}$ with $\mathbb{E}|X_{ij}|^2 = \sigma^2 < \infty$.
- Independence on and above the diagonal.
- Consider $\boldsymbol{X}_N$ and $\boldsymbol{Y}_N$ the $N \times N$ hermitian matrices defined by

$$[\boldsymbol{X}_N]_{ij} = \begin{cases} X_{ij} & \text{if } i \le j \\ \overline{X_{ji}} & \text{if } i > j \end{cases} \qquad \text{and} \qquad \boldsymbol{Y}_N = \frac{\boldsymbol{X}_n}{\sqrt{N}}$$

Then almost surely,

$$\boxed{L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(\boldsymbol{Y}_N)} \quad \xrightarrow[N\to\infty]{\mathrm{w}} \quad \mathbb{P}_{\mathrm{sc},\sigma^2}}$$

Remarks

- As a consequence of Wigner's theorem: $\boxed{\dfrac{\#\{\lambda_i \in [a,b]\}}{N} \xrightarrow[N\to\infty]{a.s.} \int_a^b \mathbb{P}_{\mathrm{sc},\sigma^2}(dx)}$.

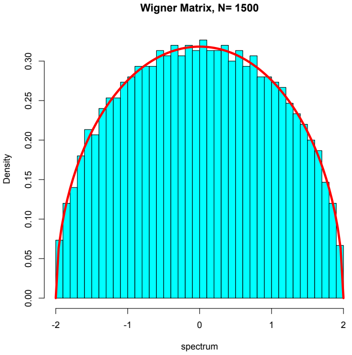

Figure: The distribution of $\mathbf{Y}_N$'s eigenvalues follows the semi-circular density

.

Additional results

Convergence of $\lambda_{\min}$ and $\lambda_{\max}$

If $\mathbb{E}|X_{ij}|^4 < \infty$ then $\boxed{\lambda_{\max}(\boldsymbol{Y}_N) \xrightarrow[N\to\infty]{a.s.} 2\sigma}$ and $\boxed{\lambda_{\min}(\boldsymbol{Y}_N) \xrightarrow[N\to\infty]{a.s.} -2\sigma}$

Fluctuations of linear statistics

let f be $C^4(\mathbb{R})$ then

$$\sum_{i=1}^{N} f(\lambda_i(\boldsymbol{Y}_N)) - N\int_{\mathbb{R}} f(x)\mathbb{P}_{\mathrm{sc}}(dx) \quad \xrightarrow[N\to\infty]{\mathcal{D}} \quad Z \sim \mathcal{N}(\beta(f), \Theta^2(f))$$

Notice the normalization + exact expression of $\beta(f)$ and $\Theta^2(f)$ complicated - cf. book by Bai and Silverstein.

Fluctuations of $\lambda_{\max}$

Let $X_{ij} \sim \mathcal{N}(0,1)$ $(i \leq j)$ then

$$N^{2/3}\left(\lambda_{\max}(\boldsymbol{Y}_N) - 2\right) \xrightarrow[N\to\infty]{\mathcal{D}} \mathbb{P}_{TW}$$

The distribution $\mathbb{P}_{TW}$ is **Tracy-Widom** distribution, hard to describe - cf. book by Anderson, Guionnet, Zeitouni.

A heuristic on the normalization $N^{2/3}$

- By Wigner's theorem, $\boxed{\frac{\#\{\lambda_i > 2-\varepsilon\}}{N} \longrightarrow \int_{2-\varepsilon}^{2} \frac{\sqrt{(4-x^2)}}{2\pi}\, dx}$
- Hence "for small ε",

$$\begin{aligned} \#\{\lambda_i > 2-\varepsilon\} &\approx N \int_{2-\varepsilon}^{2} \frac{\sqrt{(2-x)(2+x)}}{2\pi}\, dx \\ &\approx N \frac{2}{2\pi} \int_{2-\varepsilon}^{2} \sqrt{2-x}\, dx \quad = \quad \frac{N}{\pi} \varepsilon^{3/2} \end{aligned}$$

- To have **finitely many values** in $(2-\varepsilon, \infty)$, we want $\boxed{\#\{\lambda_i > 2-\varepsilon\} = \mathcal{O}(1)}$
- We choose $\varepsilon = cN^{-2/3}$ so that $\boxed{N\varepsilon^{3/2} = \mathcal{O}(1)}$ and

$$\#\{\lambda_i > 2 - cN^{-2/3}\} \quad = \quad \#\{N^{2/3}(\lambda_i - 2) > c\} \quad = \quad \mathcal{O}(1)$$

- This suggests to study the fluctuations of $\boxed{N^{2/3}\,(\lambda_{\max} - 2)}$
- The $N^{2/3}$ normalization is strongly associated to the $\sqrt{x}$-behaviour of the density at the corresponding edge

Weak convergence - the moment method

- Under proper assumptions (ex: compact support), the moments of distribution μ

$$m_k = \int x^k \mu(dx)$$

fully characterize the distribution μ.

- Given (μ_N) and μ, the convergence of the moments

$$m_k^{(N)} = \int x^k \mu_N(dx) \xrightarrow[N\to\infty]{} m_k = \int x^k \mu(dx)$$

characterizes the (weak) convergence of the measures $\mu_N \xrightarrow[N\to\infty]{w} \mu$.

The moment method ..

.. aims at proving that

$$\boxed{m_k^{(N)} \xrightarrow[N\to\infty]{} m^k}$$

Moments of the spectral measure

Recall that $L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$ and

$$m_k^{(N)} = \int x^k d\, L_N(dx) = \frac{1}{N}\sum_{i=1}^{N} \lambda_i^k = \frac{1}{N}\mathrm{Trace}(\boldsymbol{Y}_N^k)$$

Outline of the proof

1. **compute the moments** of the semi-circular distribution:

$$\int_{-2}^{2} \lambda^k \frac{\sqrt{4-\lambda^2}}{2\pi}\, d\lambda \quad = \begin{cases} \frac{1}{\ell+1}\binom{2\ell}{\ell} & \text{if } k = 2\ell\ , \\ 0 & \text{if } k = 2\ell+1 \end{cases}$$

2. **compute (the expectation of) the asymptotic moments** of the spectral distribution

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$$

that is

$$\begin{aligned} \mathbb{E}\, m_k^{(N)} &= \frac{1}{N}\mathbb{E}\,\mathrm{Trace}\,\boldsymbol{Y}_N^k \\ &= \frac{1}{N^{1+\frac{k}{2}}} \sum_{i_1,\cdots,i_k=1}^{N} \mathbb{E} X_{i_1 i_2} X_{i_2 i_3} \cdots X_{i_k i_1} \end{aligned}$$

3. **prove that**

$$\mathbb{E} m_k^{(N)} \xrightarrow[N\to\infty]{} \begin{cases} \frac{1}{\ell+1}\binom{2\ell}{\ell} & \text{if } k = 2\ell\ , \\ 0 & \text{if } k = 2\ell+1 \end{cases}$$

$\Rightarrow$ Computation of empirical moments heavily relies on (sometimes difficult) **combinatorics**.

4. **Prove some concentration:** $m_k^{(N)} - \mathbb{E} m_k^{(N)} \xrightarrow[N\to\infty]{} 0$

Wigner's theorem: computation of the moments I

$$\boxed{\mathbb{E}\, m_p^{(N)} = \frac{1}{N}\mathbb{E}\sum_{i=1}^{N}\lambda_i^p = \frac{1}{N}\mathbb{E}\,\mathrm{Trace}\left(\frac{\mathbf{X}_N}{\sqrt{N}}\right)^p}$$

- Simple but **very important formula** because it establishes the connection between the **eigenvalues** and the **entries** of $\mathbf{X}_N$.

$$\frac{1}{N}\mathbb{E}\,\mathrm{Trace}\left(\frac{\mathbf{X}_N}{\sqrt{N}}\right)^p = \frac{1}{N^{1+\frac{p}{2}}}\underbrace{\sum_{i_1,\cdots,i_p=1}^{N}\mathbb{E}X_{i_1i_2}X_{i_2i_3}\cdots X_{i_pi_1}}_{\text{a priori } N^p \text{ terms}}$$

- Effectively, **much less contributing terms**: $\boxed{\sim N^{1+\frac{p}{2}} \text{ contributing terms}}$
- For example

$$\mathbb{E}X_{11}X_{12}X_{21} = \mathbb{E}X_{11}X_{12}^2 = \mathbb{E}X_{11}\mathbb{E}X_{12}^2 = 0\,.$$

  or

$$\mathbb{E}X_{11}\cdots X_{11} = \mathbb{E}X_{11}^p \quad \leftrightarrow \quad N \text{ terms}$$

- If p is odd then

$$\frac{1}{N}\mathbb{E}\,\mathrm{Trace}\left(\frac{\mathbf{X}_N}{\sqrt{N}}\right)^p \xrightarrow[N\to\infty]{} 0$$

  (easy argument for symmetric entries).

Wigner's theorem: computation of the moments II

$$\boxed{\frac{1}{N}\mathbb{E}\,\mathrm{Trace}\left(\frac{\mathbf{X}_N}{\sqrt{N}}\right)^p = \frac{1}{N^{1+\frac{p}{2}}}\sum_{i_1,\cdots,i_p=1}^{N}\mathbb{E}X_{i_1i_2}X_{i_2i_3}\cdots X_{i_pi_1}}$$

- **Fact:** the only contributing terms are those for which p is **even** and each random variable appears exactly twice

  $$\mathbb{E}X_{12}X_{23}X_{34}X_{43}X_{32}X_{21} = \mathbb{E}X_{12}^2X_{23}^2X_{34}^2$$

  so there are exactly $\frac{p}{2}+1$ degrees of freedom.
- For each configuration, say

  $$\mathbb{E}X_{12}X_{21}X_{13}X_{31}$$

  we count the number of terms with the same pattern obtained by permutations such as

  $$\mathbb{E}X_{16}X_{61}X_{15}X_{51}\,,\quad \mathbb{E}X_{24}X_{42}X_{28}X_{82}\,,\quad \mathbb{E}X_{n2}X_{2n}X_{n3}X_{3n}\,.$$

  There are $\boxed{N\times(N-1)\times\cdots\times(N-\frac{P}{2})\sim N^{1+\frac{p}{2}}}$ terms.
- Now we need to count the number of different configurations

  $$\begin{aligned}\mathbb{E}X_{12}X_{21}X_{13}X_{31}X_{14}X_{41}\quad &\text{different from}\quad \mathbb{E}X_{12}X_{23}X_{32}X_{21}X_{14}X_{41}\\ &\text{different from}\quad \mathbb{E}X_{12}X_{23}X_{34}X_{43}X_{32}X_{21}\end{aligned}$$

Wigner's theorem: computation of the moments III

$$\boxed{\frac{1}{N}\mathbb{E}\,\mathrm{Trace}\left(\frac{\mathbf{X}_N}{\sqrt{N}}\right)^p = \frac{1}{N^{1+\frac{p}{2}}} \times N^{1+\frac{p}{2}} \times \{\text{number of different configurations}\} + o(1)}$$

- There is a one-to-one correspondence between the number of p-configurations and the number of **Dyck paths** of length p.

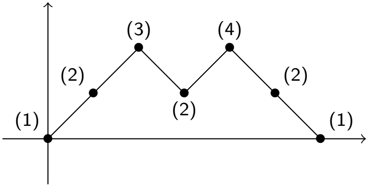

Figure: Dyck path of length 6 associated to $\mathbb{E}X_{12}X_{23}X_{32}X_{24}X_{42}X_{21}$

Wigner's theorem: computation of the moments III

$$\boxed{\frac{1}{N}\mathbb{E}\,\mathrm{Trace}\left(\frac{\mathbf{X}_N}{\sqrt{N}}\right)^p = \frac{1}{N^{1+\frac{p}{2}}} \times N^{1+\frac{p}{2}} \times \{\text{number of different configurations}\} + o(1)}$$

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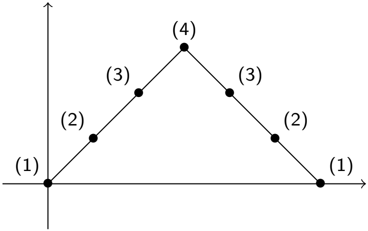

Figure: Dyck path of length 6 associated to $\mathbb{E}X_{12}X_{23}X_{34}X_{43}X_{32}X_{21}$

Wigner's theorem: computation of the moments III

$$\boxed{\frac{1}{N}\mathbb{E}\,\mathrm{Trace}\left(\frac{\mathbf{X}_N}{\sqrt{N}}\right)^p = \frac{1}{N^{1+\frac{p}{2}}} \times N^{1+\frac{p}{2}} \times \{\text{number of different configurations}\} + o(1)}$$

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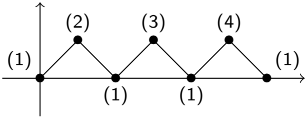

Figure: Dyck path associated to $\mathbb{E}X_{12}X_{21}X_{13}X_{31}X_{14}X_{41}$

Wigner's theorem: computation of the moments III

$$\boxed{\frac{1}{N}\mathbb{E}\,\mathrm{Trace}\left(\frac{\mathbf{X}_N}{\sqrt{N}}\right)^p = \frac{1}{N^{1+\frac{p}{2}}} \times N^{1+\frac{p}{2}} \times \{\text{number of different configurations}\} + o(1)}$$

- There is a one-to-one correspondence between the number of p-configurations and the number of **Dyck paths** of length p.

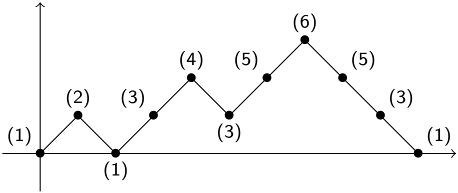

Figure: Dyck path of length 10 associated to $\mathbb{E}X_{12}X_{21}X_{13}X_{34}X_{43}X_{35}X_{56}X_{65}X_{53}X_{31}$

Wigner's theorem: computation of the moments III

$$\boxed{\frac{1}{N}\mathbb{E}\,\mathrm{Trace}\left(\frac{\mathbf{X}_N}{\sqrt{N}}\right)^p = \frac{1}{N^{1+\frac{p}{2}}} \times N^{1+\frac{p}{2}} \times \{\text{number of different configurations}\} + o(1)}$$

- There is a one-to-one correspondence between the number of p-configurations and the number of **Dyck paths** of length p.

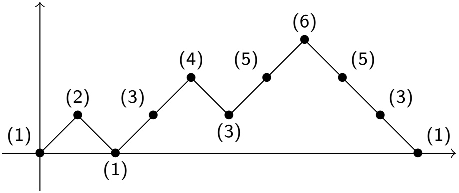

Figure: Dyck path associated to $\mathbb{E}X_{12}X_{21}X_{13}X_{34}X_{43}X_{35}X_{56}X_{65}X_{53}X_{31}$

- The total number of Dyck paths is well known:

$$\boxed{\#\{\text{Dyck paths}\} = \frac{1}{1+k}\binom{2k}{k}} \quad (p = 2k)$$

and corresponds to the moments of the semi-circle distribution.

Wigner's theorem: concentration

We can prove (by combinatorial arguments) that

$$\boxed{\mathrm{var}\left(\frac{1}{N}\int x^p L_N(dx)\right) = \mathcal{O}\left(\frac{1}{N^2}\right)}$$

- Hence by Borel-Cantelli,

$$\forall \boldsymbol{p} \geq \mathbf{1}\,, \quad \textbf{(a.s.)} \quad \frac{1}{N}\int x^p L_N(dx) \xrightarrow[N\to\infty]{} \begin{cases} \frac{1}{1+k}\binom{2k}{k} & p = 2k \\ 0 & p = 2k+1 \end{cases} ,$$

- From which we deduce that

$$\textbf{(a.s.)}\,, \quad \forall \boldsymbol{p} \geq \mathbf{1} \quad \frac{1}{N}\int x^p L_N(dx) \xrightarrow[N\to\infty]{} \begin{cases} \frac{1}{1+k}\binom{2k}{k} & p = 2k \\ 0 & p = 2k+1 \end{cases} ,$$

- and conclude

$$(\mathrm{a.s.})\,, \quad L_N = \frac{1}{N}\sum_i \delta_{\lambda_i} \xrightarrow[N\to\infty]{w} \frac{\sqrt{(4-x^2)_+}}{2\pi}\,dx$$

Large Covariance Matrices I

The model

- Consider a $N \times n$ matrix $\boldsymbol{X}_n$ with i.i.d. entries

$$\mathbb{E}X_{ij} = 0 \;, \quad \mathbb{E}|X_{ij}|^2 = \sigma^2 \;.$$

Matrix $\boldsymbol{X}_n$ is a n-sample of N-dimensional vectors:

$$\boldsymbol{X}_n = [\boldsymbol{x}_1 \;\; \cdots \;\; \boldsymbol{x}_n] \quad \text{with} \quad \mathbb{E}\,\boldsymbol{x}_1\boldsymbol{x}_1^* = \sigma^2 \mathbf{I}_N \;.$$

Objective

- to describe **the limiting spectrum** of $\frac{1}{n}\boldsymbol{X}_n\boldsymbol{X}_n^*$ as

$$\frac{N}{n} \xrightarrow[n\to\infty]{} c \in (0,\infty) \;.$$

i.e. dimensions of matrix $\boldsymbol{X}_n$ are **of the same order**.

The standard statistical case $N << n$ (small data, large sample)

Assume N fixed and $n \to \infty$. Since

$$\mathbb{E}\, \boldsymbol{x}_1 \boldsymbol{x}_1^* = \sigma^2 \mathbf{I}_N \ ,$$

L.L.N implies

$$\frac{1}{n} \boldsymbol{X}_n \boldsymbol{X}_n^* = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i^* \quad \xrightarrow[n\to\infty]{a.s.} \quad \sigma^2 \mathbf{I}_N$$

In particular,

- all the eigenvalues of $\frac{1}{n}\boldsymbol{X}_n \boldsymbol{X}_n^*$ converge to σ^2,
- equivalently, the spectral measure of $\frac{1}{n}\boldsymbol{X}_n \boldsymbol{X}_n^*$ converges to δ_{σ^2}.

A priori observation # 1

> If the ratio of dimensions $c \searrow 0$, then the spectral measure should look like a Dirac measure at point σ^2.

Large Covariance Matrices III

The case where $c > 1$

Recall that $\boldsymbol{X}_n$ is $N \times n$ matrix and $c = \lim \frac{N}{n}$.

If $N > n$, then $\frac{1}{n}\boldsymbol{X}_n\boldsymbol{X}_n^*$ is **rank-defficient** and has rank n;

- in this case, eigenvalue 0 has multiplicity $N - n$ and the spectral measure writes:

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i} = \frac{1}{N}\sum_{i=1}^{n} \delta_{\lambda_i} + \frac{N-n}{N}\delta_0$$

- The limiting spectral measure of L_N **necessarily features a Dirac measure at 0**:

$$\frac{N-n}{N}\delta_0 \longrightarrow \left(1 - \frac{1}{c}\right)\delta_0 \quad \text{as} \quad \frac{N}{n} \to c\ .$$

A priori observation #2

If $c > 1$, then the limiting spectral measure will feature a Dirac measure at 0 with weight $1 - \frac{1}{c}$.

Marčenko-Pastur's theorem

Theorem

- Consider a $N \times n$ matrix $\boldsymbol{X}_n$ with i.i.d. entries

$$\mathbb{E}X_{ij} = 0 \ , \quad \mathbb{E}|X_{ij}|^2 = \sigma^2 \ .$$

with N and n **of the same order** and L_N the **spectral measure** of $\frac{1}{n}\boldsymbol{X}_n\boldsymbol{X}_n^*$:

$$c_n \triangleq \frac{N}{n} \xrightarrow[n\to\infty]{} c \in (0,\infty) \ , \quad L_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i} \ , \quad \lambda_i = \lambda_i\left(\frac{1}{n}\boldsymbol{X}_n\boldsymbol{X}_n^*\right)$$

- Then **almost surely** (= for almost every realization)

$$L_N \xrightarrow[N,n\to\infty]{} \mathbb{P}_{\check{\mathrm{M}}\mathrm{P}} \quad \text{in distribution}$$

where $\mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}$ is **Marčenko-Pastur** distribution:

$$\begin{array}{rcl} \mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}(dx) & = & \left(1-\frac{1}{c}\right)_+ \delta_0(dx) + \frac{\sqrt{[(\lambda^+ - x)(x-\lambda^-)]_+}}{2\pi\sigma^2 xc}\,dx \\ \text{with} & & \left\{ \begin{array}{rcl} \lambda^- & = & \sigma^2(1-\sqrt{c})^2 \\ \lambda^+ & = & \sigma^2(1+\sqrt{c})^2 \end{array} \right. \end{array}$$

Simulations vs MP distribution

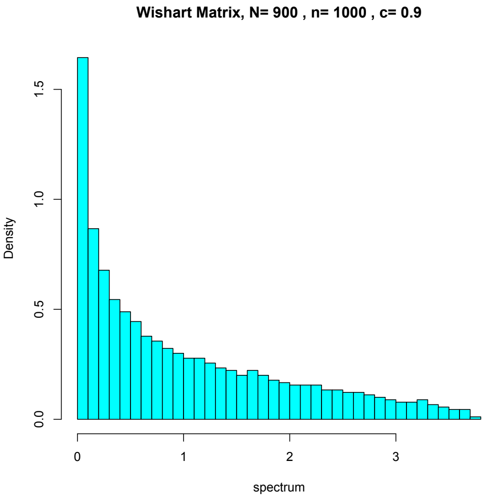

Figure: Histogram of $\frac{1}{n}\boldsymbol{X}_n\boldsymbol{X}_n^*$, $\sigma^2 = 1$

Simulations vs M̌P distribution

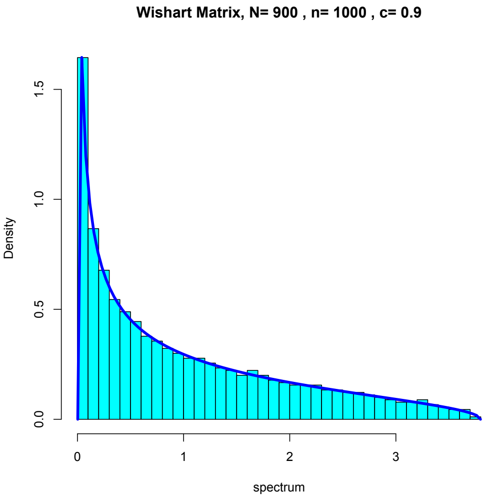

Figure: Marčenko-Pastur distribution for $c = 0.9$

Simulations vs MP distribution

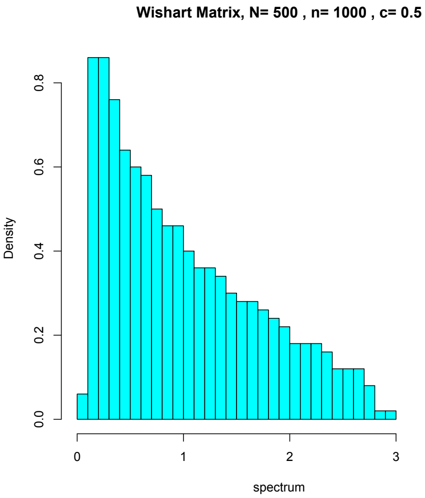

Figure: Histogram of $\frac{1}{n}\boldsymbol{X}_n\boldsymbol{X}_n^*$, $\sigma^2 = 1$

Simulations vs M̌P distribution

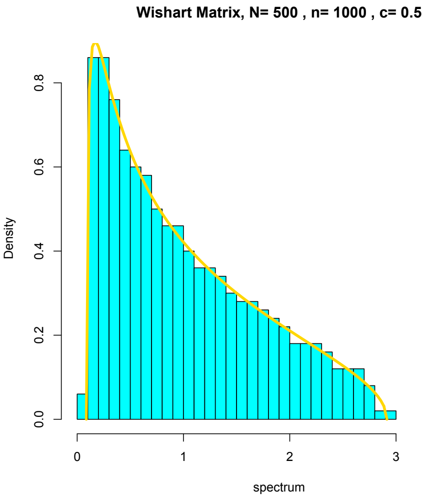

Figure: Marčenko-Pastur distribution for $c = 0.5$

Simulations vs MP distribution

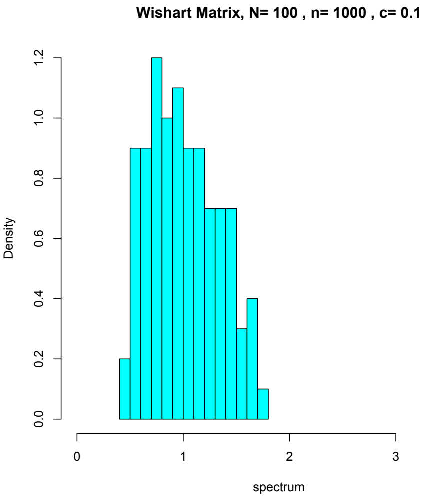

Figure: Histogram of $\frac{1}{n}\boldsymbol{X}_n\boldsymbol{X}_n^*$, $\sigma^2 = 1$

Simulations vs MP distribution

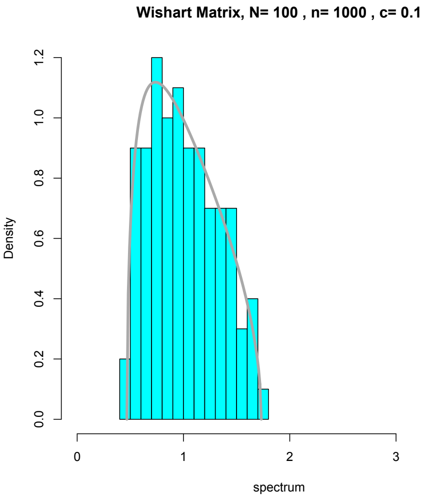

Figure: Marčenko-Pastur distribution for $c = 0.1$

Simulations vs MP distribution

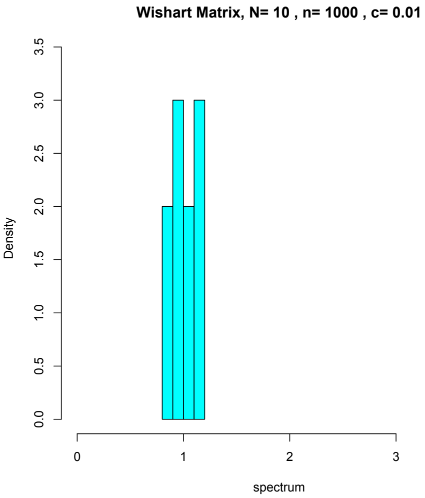

Figure: Histogram of $\frac{1}{n}\boldsymbol{X}_n\boldsymbol{X}_n^*$, $\sigma^2 = 1$

Simulations vs MP distribution

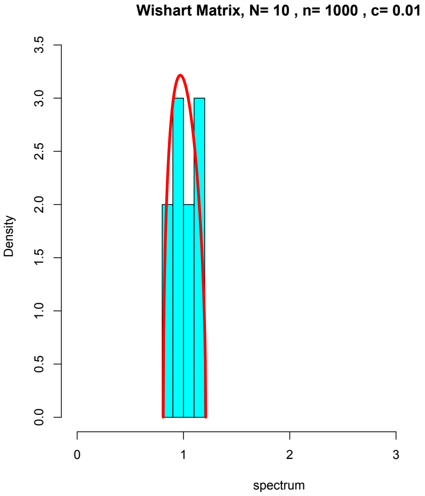

Figure: Marčenko-Pastur distribution for $c = 0.01$

Strategy of proof: The Stieltjes Transform

Given a probability measure $\mathbb{P}$, its **Stieltjes transform** is a function

$$g(z) = \int_{\mathbb{R}} \frac{\mathbb{P}(d\lambda)}{\lambda - z} , \quad z \in \mathbb{C}^+ , \qquad (\text{Notation: } g = ST(\mathbb{P}))$$

with inverse formulas

$$\mathbb{P}(a,b) \quad = \quad \frac{1}{\pi} \lim_{y \downarrow 0} \Im \int_a^b g(x + \mathbf{i}y)\, dx , \quad \text{if } \mathbb{P}\{a\} = \mathbb{P}\{b\} = 0$$

Example

- Spectral measure:

$$\mathbb{P} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i} \quad \Rightarrow \quad g(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - z}$$

Proposition

Let $\boldsymbol{Y}_N$ a hermitian matrix with spectral measure L_N then $g = ST(L_N)$ satisfies

$$\boxed{g(z) = \frac{1}{N} \mathrm{Trace}\, (\boldsymbol{Y}_N - zI_N)^{-1}}$$

Strategy of proof II: the ST satisfies an equation

Recall definition of the **Stieltjes transform** g_n:

$$g_n(z) \quad = \quad \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i - z} \quad = \quad \frac{1}{N}\mathrm{Trace}\left(\frac{1}{n}\mathbf{X}_N\mathbf{X}_N^* - z\mathbf{I}_N\right)^{-1} \; .$$

1. **[Concentration]** we first prove that

$$\boxed{\mathrm{var}(g_n(z)) = \mathcal{O}\left(\frac{1}{n^2}\right)}$$

   and hence focus on $\mathbb{E}g_n(z)$.

2. After algebraic manipulations and probabilistic arguments, we prove that

$$\mathbb{E}\, g_n(z) = \frac{1}{\sigma^2(1-c_n) - z - z\sigma^2 c_n \mathbb{E}\, g_n(z)} + \varepsilon_n(z) \quad \text{with} \quad \varepsilon_n(z) \xrightarrow[N,n\to\infty]{} 0 \; .$$

3. **[Stability]** By stability of Marčenko-Pastur's equation, $\mathbb{E}\, g_n$ converges to a function $\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}$ which satisfies **the fixed point equation**:

$$\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z) = \frac{1}{\sigma^2(1-c) - z - z\sigma^2 c\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)}$$

4. We identify $\boxed{\mathbb{P}_{\check{\mathrm{M}}\mathrm{P}} = (\text{Stieltjes Transform})^{-1}(\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}})}$.

Elements of proof I

Notations

We introduce the column (normalized) vectors of matrix $\boldsymbol{X}_N$:

$$\frac{1}{n}\boldsymbol{X}_N\boldsymbol{X}_N^* = \sum_{i=1}^{n} \boldsymbol{y}_i\boldsymbol{y}_i^* \quad \text{with} \quad \frac{\boldsymbol{X}_N}{\sqrt{N}} = [\boldsymbol{y}_1, \cdots, \boldsymbol{y}_n] \,.$$

The resolvent writes

$$Q = \left(\sum_{i=1}^{n} \boldsymbol{y}_i\boldsymbol{y}_i^* - zI_N\right)^{-1} \quad \text{and we introduce} \quad Q_i = \left(\sum_{j\neq i} \boldsymbol{y}_j\boldsymbol{y}_j^* - zI_N\right)^{-1} \,.$$

Elements of proof II

Rank-one perturbations

- Sherman-Morrison identity

$$Q\boldsymbol{y}_i\boldsymbol{y}_i^* = \frac{Q_i\boldsymbol{y}_i\boldsymbol{y}_i^*}{1+\boldsymbol{y}_i^*Q_i\boldsymbol{y}_i}$$

- Asymptotic rank-one perturbation: let $\boldsymbol{u} \in \mathbb{R}^N$ and $\boldsymbol{u}\boldsymbol{u}^*$ a rank-one matrix then

$$\left|\frac{1}{N}\mathrm{Trace}(\mathbf{A}+\vec{u}\vec{u}^*-z\mathbf{I}_N)^{-1}-\frac{1}{N}\mathrm{Trace}(\mathbf{A}-z\mathbf{I}_N)^{-1}\right| \leq \frac{1}{N\Im(z)}$$

In particular

$$\frac{1}{N}\mathrm{Trace}\, Q - \frac{1}{N}\mathrm{Trace}\, Q_i \xrightarrow[N\to\infty]{} 0\,.$$

Elements of proof III: exact computations

Massaging the resolvent

$$
\begin{aligned}
Q\left(\frac{1}{n}\boldsymbol{X}\boldsymbol{X}^* - zI_N\right) = I_N \quad &\Leftrightarrow \quad Q\sum_i \boldsymbol{y}_i\boldsymbol{y}_i^* - zQ = I_N \\
&\Leftrightarrow \quad zQ = -I_N + Q\sum_i \boldsymbol{y}_i\boldsymbol{y}_i^* \\
&\Leftrightarrow \quad zQ = -I_N + \sum_{i=1}^n \frac{Q_i \sum_i \boldsymbol{y}_i\boldsymbol{y}_i^*}{1+\boldsymbol{y}_i^* Q_i \boldsymbol{y}_i} \\
&\Leftrightarrow \quad zQ = -I_N + \sum_{i=1}^n \left(1 - \frac{1}{1+\boldsymbol{y}_i^* Q_i \boldsymbol{y}_i}\right)
\end{aligned}
$$

Going back to the Stieltjes transform $g_n(z) = \frac{1}{N}\mathrm{Trace}\, Q$

Taking $\frac{1}{N}\mathrm{Trace}\{\cdot\} + \mathbb{E}\{\cdot\}$ yields

$$
\begin{aligned}
z\mathbb{E}g_n(z) &= -1 + \frac{1}{N}\sum_{i=1}^n \left(1 - \mathbb{E}\frac{1}{1+\boldsymbol{y}_i^* Q_i \boldsymbol{y}_i}\right) \\
&= -1 + \frac{1}{c_N}\left(1 - \mathbb{E}\frac{1}{1+\boldsymbol{y}_1^* Q_1 \boldsymbol{y}_1}\right)
\end{aligned}
$$

Elements of proof IV: approximate equation

- We can prove that

$$z\mathbb{E}g_n(z) = -1+\frac{1}{c_N}\left(1-\mathbb{E}\frac{1}{1+\boldsymbol{y}_1^*Q_1\boldsymbol{y}_1}\right) = -1+\frac{1}{c_N}\left(1-\frac{1}{1+\mathbb{E}\boldsymbol{y}_1^*Q_1\boldsymbol{y}_1}\right)+o(1)\,.$$

  Now

$$\mathbb{E}\boldsymbol{y}_1^*Q_1\boldsymbol{y}_1 = \frac{\sigma^2}{n}\mathbb{E}\mathrm{Trace}\, Q_1 \simeq \frac{\sigma^2}{n}\mathbb{E}\mathrm{Trace}\, Q = c_N\sigma^2\mathbb{E}g_n(z)$$

- We end up with the following equation

$$z\mathbb{E}g_n(z) = -1+\frac{1}{c_N}\left(1-\frac{1}{1+c_n\sigma^2\mathbb{E}g_n(z)}\right)+o(1)$$

  which writes

$$\mathbb{E}\, g_n(z) = \frac{1}{\sigma^2(1-c_n)-z-z\sigma^2 c_n\mathbb{E}\, g_n(z)}+o(1)$$

  and corresponds to the **approximate version** of the equation

$$\boxed{\mathbf{g}_{\check{\mathrm{MP}}}(z) = \frac{1}{\sigma^2(1-c)-z-z\sigma^2 c\,\mathbf{g}_{\check{\mathrm{MP}}}(z)}}$$

Solving the limiting equation

Explicit Stieltjes transform

- Given the second-degree polynomial $\boxed{zc\sigma^2 \mathbf{g}_{\check{\text{MP}}}^2 + [z - \sigma^2(1-c)]\mathbf{g}_{\check{\text{MP}}} + 1 = 0}$,
- an explicit solution is given by

$$\boxed{\mathbf{g}_{\check{\text{MP}}}(z) = \frac{-(z + \sigma^2(c-1)) + \sqrt{(z-\lambda^+)(z-\lambda^-)}}{2zc\sigma^2}}$$

with $a\lambda^- = \sigma^2(1-\sqrt{c})^2$ and $\lambda^+ = \sigma^2(1+\sqrt{c})^2$ and where $\sqrt{(\cdot)}$ refers to the appropriate branch of the square root function.

Marčenko-Pastur's distribution

The inverse formula

$$\mathbb{P}_{\check{\text{MP}}}[a,b] = \frac{1}{\pi} \lim_{y \downarrow 0} \Im \int_a^b \mathbf{g}_{\check{\text{MP}}}(x + \mathbf{i}y)\, dx$$

can be used to find:

$$\boxed{\mathbb{P}_{\check{\text{MP}}}(dx) \;=\; \left(1 - \frac{1}{c}\right)_+ \delta_0(dx) + \frac{\sqrt{[(\lambda^+ - x)(x - \lambda_-)]_+}}{2\pi\sigma^2 xc}\, dx}$$

Stability of the canonical equation

Theorem (stability)

- The canonical equation is **stable**: if

$$z c \sigma^2 \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}^2 + [z - \sigma^2(1-c)]\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}} + 1 \quad = \quad 0$$

  and

$$z c_\delta \sigma^2 \mathbf{g}_\delta^2 + [z - \sigma^2(1-c_\delta)]\boldsymbol{g}_\delta + 1 \quad = \quad \delta$$

  then

$$\boxed{|\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}} - \boldsymbol{g}_\delta| = \mathcal{O}(|\delta| + |c - c_\delta|)}$$

- In particular, since

$$\mathbb{E} g_n(z) \quad = \quad \frac{1}{\sigma^2(1-c_n) - z - z\sigma^2 c_n \mathbb{E} g_n(z)} + \varepsilon_n$$

  or equivalently

$$z c_N \sigma^2 (\mathbb{E} g_n)^2 + [z - \sigma^2(1-c_N)]\mathbb{E} g_n + 1 = \varepsilon_n$$

  we have $\boxed{|\mathbb{E} g_n - \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}| = \mathcal{O}(|\varepsilon_n| + |c - c_N|)}$ and

$$g_n(z) \xrightarrow[N,n\to\infty]{a.s.} \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)$$

Marčenko-Pastur's Theorem: Summary

- Consider the model $\frac{1}{n}\mathbf{X}_N\mathbf{X}_N^*$, then its spectral measure satisfies:

$$\boxed{\text{a. s.}\quad L_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i} \xrightarrow[N,n\to\infty]{\mathcal{L}} \mathbb{P}_{\check{\mathrm{MP}}}\ .}$$

- Instead of directly working on L_N, we consider its **Stieltjes tranform**

$$g_n(z) = \frac{1}{N}\mathrm{Trace}\left(\frac{1}{n}\mathbf{X}_N\mathbf{X}_n^* - z\mathbf{I}_N\right)^{-1}\ ,$$

  then prove that it **concentrates near its expectation**

- and satisfies the **approximate fixed-point** equation

$$\boxed{\mathbb{E}\,g_n(z) \approx \frac{1}{\sigma^2(1-c_n) - z - z\sigma^2 c_n \mathbb{E}\,g_n(z)}}$$

  and that it converges to the solution $\mathbf{g}_{\check{\mathrm{MP}}}$ of the **canonical equation**

$$\boxed{\mathbf{g}_{\check{\mathrm{MP}}}(z) = \frac{1}{\sigma^2(1-c) - z - z\sigma^2 c\,\mathbf{g}_{\check{\mathrm{MP}}}(z)}}$$

- Computing explicitely $\mathbf{g}_{\check{\mathrm{MP}}}$ and inverting it yields finally the formula for $\mathbb{P}_{\check{\mathrm{MP}}}$.

Large covariance matrices

The model

- Consider a $N \times n$ matrix $\boldsymbol{X}_n$ with i.i.d. entries $\mathbb{E}X_{ij} = 0, \mathbb{E}|X_{ij}|^2 = 1$.
- Let $\mathbf{R}_N$ be a **deterministic** $N \times N$ nonnegative definite hermitian matrix.
- Consider

$$\boldsymbol{Y}_n = \mathbf{R}_N^{1/2} \boldsymbol{X}_n \ .$$

  with $\boldsymbol{R}_N$ the (deterministic) **Population covariance matrix**.
- Matrix $\boldsymbol{Y}_n$ is a n-sample of N-dimensional vectors:

$$\boldsymbol{Y}_n = [\boldsymbol{y}_1 \ \cdots \ \boldsymbol{y}_n] \quad \text{with} \quad \boldsymbol{y}_1 = \mathbf{R}_N^{1/2} \boldsymbol{x}_1 \quad \text{and} \quad \mathbb{E}\, \boldsymbol{y}_1 \boldsymbol{y}_1^* = \mathbf{R}_N \ .$$

Theorem

- The spectral measure of $\boldsymbol{Y}_n$ converges toward

$$\boxed{\mathbb{P}_\infty = \mathbb{P}^{\mathbf{R}} \boxtimes \mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}}$$

  the **free multiplicative convolution** of the limit of the spectral measure of $\boldsymbol{R}_N$ and MP.

Remark (small data, large sample)

- If N fixed and $n \to \infty$ then $\boxed{\frac{1}{n} \boldsymbol{Y}_n \boldsymbol{Y}_n^* \longrightarrow \mathbf{R}_N}$

Simulations

- Consider the distribution

$$\mathbb{P}^{\mathbf{R}} = \frac{1}{3}\delta_1 + \frac{1}{3}\delta_3 + \frac{1}{3}\delta_7$$

corresponding to a covariance matrix

$$\mathbf{R}_N = \operatorname{diag}(1, 3, 7)$$

each with multiplicity $\approx \frac{N}{3}$.

- We plot hereafter

$$\mathbb{P}_\infty = \mathbb{P}^{\mathbf{R}} \boxtimes \mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}$$

for different values of c.

$$\boxed{\mathbf{t}(z) = \frac{\mathbf{1}}{\mathbf{3}}\left\{\frac{1}{(1-c)\boldsymbol{\lambda}_1 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_1} + \frac{1}{(1-c)\boldsymbol{\lambda}_2 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_2} + \frac{1}{(1-c)\boldsymbol{\lambda}_3 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_3}\right\}}$$

Simulations

- Consider the distribution

$$\mathbb{P}^{\mathbf{R}} = \frac{1}{3}\delta_1 + \frac{1}{3}\delta_3 + \frac{1}{3}\delta_7$$

corresponding to a covariance matrix

$$\mathbf{R}_N = \mathrm{diag}(1, 3, 7)$$

each with multiplicity $\approx \frac{N}{3}$.

- We plot hereafter

$$\mathbb{P}_\infty = \mathbb{P}^{\mathbf{R}} \boxtimes \mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}$$

for different values of c.

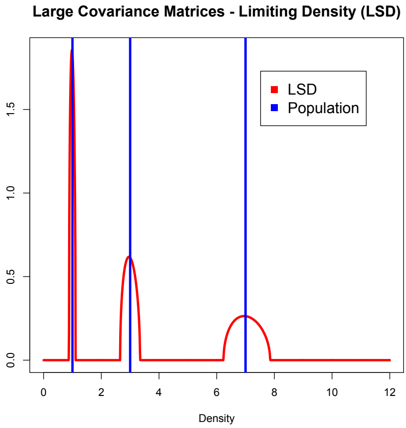

Figure: Plot of the Limiting Spectral Measure for $c = 0.01$

$$\boxed{\mathbf{t}(z) = \frac{\mathbf{1}}{\mathbf{3}}\left\{\frac{1}{(1-c)\boldsymbol{\lambda}_1 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_1} + \frac{1}{(1-c)\boldsymbol{\lambda}_2 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_2} + \frac{1}{(1-c)\boldsymbol{\lambda}_3 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_3}\right\}}$$

Simulations

- Consider the distribution

$$\mathbb{P}^{\mathbf{R}} = \frac{1}{3}\delta_1 + \frac{1}{3}\delta_3 + \frac{1}{3}\delta_7$$

corresponding to a covariance matrix

$$\mathbf{R}_N = \mathrm{diag}(1,3,7)$$

each with multiplicity $\approx \frac{N}{3}$.

- We plot hereafter

$$\mathbb{P}_\infty = \mathbb{P}^{\mathbf{R}} \boxtimes \mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}$$

for different values of c.

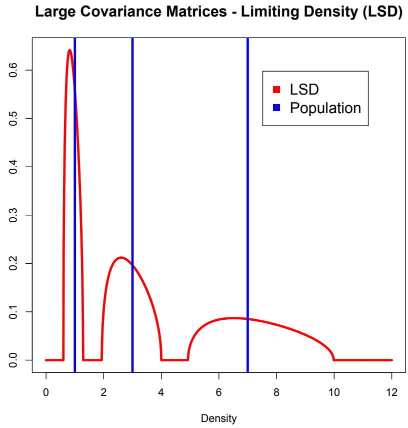

Figure: Plot of the Limiting Spectral Measure for $c = 0.1$

$$\boxed{\mathbf{t}(z) = \frac{\mathbf{1}}{\mathbf{3}}\left\{\frac{1}{(1-c)\boldsymbol{\lambda}_1 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_1} + \frac{1}{(1-c)\boldsymbol{\lambda}_2 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_2} + \frac{1}{(1-c)\boldsymbol{\lambda}_3 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_3}\right\}}$$

Simulations

- Consider the distribution

$$\mathbb{P}^{\mathbf{R}} = \frac{1}{3}\delta_1 + \frac{1}{3}\delta_3 + \frac{1}{3}\delta_7$$

corresponding to a covariance matrix

$$\mathbf{R}_N = \mathrm{diag}(1,3,7)$$

each with multiplicity $\approx \frac{N}{3}$.

- We plot hereafter

$$\mathbb{P}_\infty = \mathbb{P}^{\mathbf{R}} \boxtimes \mathbb{P}_{\check{\mathrm{MP}}}$$

for different values of c.

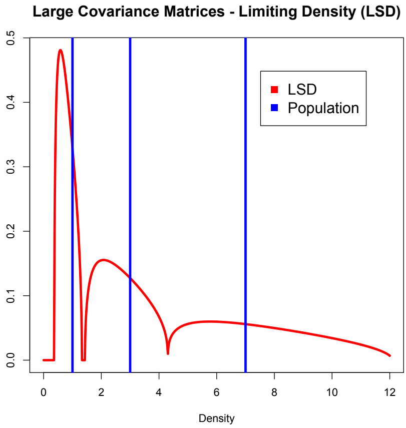

Figure: Plot of the Limiting Spectral Measure for $c = 0.25$

$$\boxed{\mathbf{t}(z) = \frac{\mathbf{1}}{\mathbf{3}}\left\{\frac{1}{(1-c)\boldsymbol{\lambda}_1 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_1} + \frac{1}{(1-c)\boldsymbol{\lambda}_2 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_2} + \frac{1}{(1-c)\boldsymbol{\lambda}_3 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_3}\right\}}$$

Simulations

- Consider the distribution

$$\mathbb{P}^{\mathbf{R}} = \frac{1}{3}\delta_1 + \frac{1}{3}\delta_3 + \frac{1}{3}\delta_7$$

corresponding to a covariance matrix

$$\mathbf{R}_N = \mathrm{diag}(1, 3, 7)$$

each with multiplicity $\approx \frac{N}{3}$.

- We plot hereafter

$$\mathbb{P}_\infty = \mathbb{P}^{\mathbf{R}} \boxtimes \mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}$$

for different values of c.

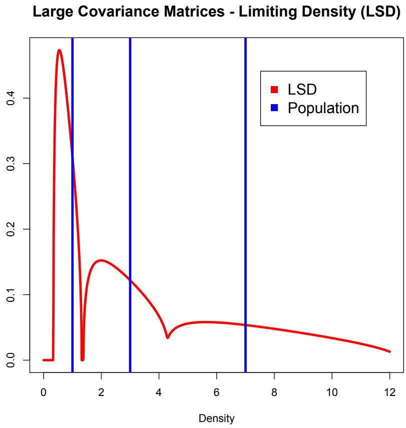

Figure: Plot of the Limiting Spectral Measure for $c = 0.275$

$$\boxed{\mathbf{t}(z) = \frac{\mathbf{1}}{\mathbf{3}}\left\{\frac{1}{(1-c)\boldsymbol{\lambda}_1 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_1} + \frac{1}{(1-c)\boldsymbol{\lambda}_2 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_2} + \frac{1}{(1-c)\boldsymbol{\lambda}_3 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_3}\right\}}$$

Simulations

- Consider the distribution
$$\mathbb{P}^{\mathbf{R}} = \frac{1}{3}\delta_1 + \frac{1}{3}\delta_3 + \frac{1}{3}\delta_7$$
corresponding to a covariance matrix
$$\mathbf{R}_N = \operatorname{diag}(1, 3, 7)$$
each with multiplicity $\approx \frac{N}{3}$.
- We plot hereafter
$$\mathbb{P}_\infty = \mathbb{P}^{\mathbf{R}} \boxtimes \mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}$$
for different values of c.

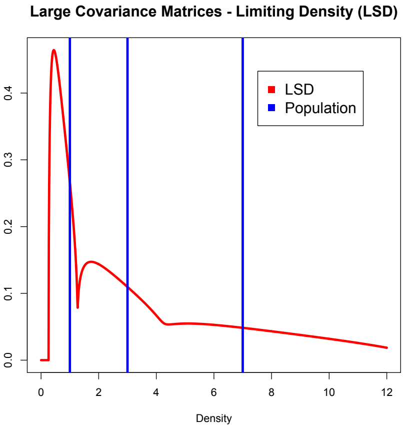

Figure: Plot of the Limiting Spectral Measure for $c = 0.35$

$$\boxed{\mathbf{t}(z) = \frac{\mathbf{1}}{\mathbf{3}}\left\{\frac{1}{(1-c)\boldsymbol{\lambda}_1 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_1} + \frac{1}{(1-c)\boldsymbol{\lambda}_2 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_2} + \frac{1}{(1-c)\boldsymbol{\lambda}_3 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_3}\right\}}$$

Simulations

- Consider the distribution

$$\mathbb{P}^{\mathbf{R}} = \frac{1}{3}\delta_1 + \frac{1}{3}\delta_3 + \frac{1}{3}\delta_7$$

corresponding to a covariance matrix

$$\mathbf{R}_N = \mathrm{diag}(1, 3, 7)$$

each with multiplicity $\approx \frac{N}{3}$.

- We plot hereafter

$$\mathbb{P}_\infty = \mathbb{P}^{\mathbf{R}} \boxtimes \mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}$$

for different values of c.

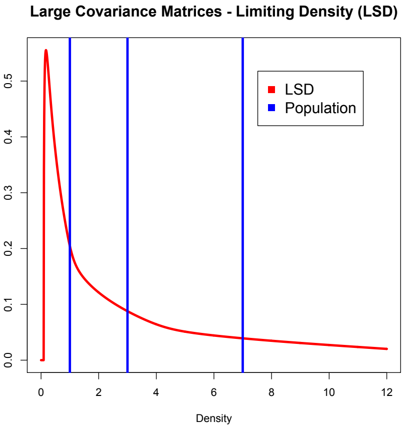

Figure: Plot of the Limiting Spectral Measure for $c = 0.6$

$$\boxed{\mathbf{t}(z) = \frac{\mathbf{1}}{\mathbf{3}}\left\{\frac{1}{(1-c)\boldsymbol{\lambda}_1 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_1} + \frac{1}{(1-c)\boldsymbol{\lambda}_2 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_2} + \frac{1}{(1-c)\boldsymbol{\lambda}_3 - z - zc\mathbf{t}(z)\boldsymbol{\lambda}_3}\right\}}$$

The largest eigenvalue in M̌P model

Let $p = p(n)$ and assume that

$$\frac{p}{n} \xrightarrow[n\to\infty]{} c \in (0,\infty)\,.$$

Consider a $p \times n$ matrix $\mathbf{X}_n$ with i.i.d. entries $\mathbb{E}X_{ij} = 0$ and $\mathbb{E}|X_{ij}|^2 = \sigma^2$, then

$$L_n\left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n^*\right) \xrightarrow[p,n\to\infty]{} \mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}$$

where $\mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}$ has support

$$\mathcal{S}_{\check{\mathrm{M}}\mathrm{P}} \;=\; \{0\} \;\cup\; \underbrace{\left[\sigma^2(1-\sqrt{c})^2\;,\;\sigma^2(1+\sqrt{c})^2\right]}_{\textbf{bulk}}$$

(remove the set $\{0\}$ if $c < 1$)

Theorem

- Let $\mathbb{E}|X_{ij}|^4 < \infty$, then:

$$\lambda_{\max}\left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n^*\right) \xrightarrow[p,n\to\infty]{a.s.} \sigma^2(1+\sqrt{c})^2\;.$$

Message: The largest eigenvalue converges to the right edge of the bulk.

Spiked Models I

Definition

Let $\mathbf{\Pi}_p$ be a **small perturbation of the identity**:

$$\mathbf{\Pi}_p = \mathbf{I}_p + \mathbf{P}_p \quad \text{where} \quad \mathbf{P}_p = \theta_1 \vec{\mathbf{u}}_1 \vec{\mathbf{u}}_1^* + \cdots + \theta_k \vec{\mathbf{u}}_k \vec{\mathbf{u}}_k^*$$

where k is **independent of the dimensions** p, n.

Consider

$$\boxed{\widetilde{\mathbf{X}}_n = \mathbf{\Pi}_p^{1/2} \mathbf{X}_n}$$

This model will be refered to as a (multiplicative) **spiked model**.

Think of $\mathbf{\Pi}_p$ as

$$\mathbf{\Pi}_p = \begin{pmatrix} 1+\theta_1 & & & & \\ & \ddots & & & \\ & & 1+\theta_k & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}$$

$\boxed{\textbf{Very important:}\ \text{The number } k \text{ of perturbations is finite}}$

Spiked Models II

Remarks

- The spiked model is a particular case of **large covariance matrix** model with

$$\mathbf{R}_p = \mathbf{I}_p + \sum_{\ell=1}^{k} \theta_\ell \vec{\mathbf{u}}_\ell \vec{\mathbf{u}}_\ell^*$$

- There are additive spiked models: $\check{\mathbf{X}}_n = \mathbf{X}_n + \mathbf{A}_n$ where $\mathbf{A}_n$ is a matrix with finite rank.

Objective

- What is the influence of $\mathbf{\Pi}_p$ over $L_N\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$?
- What is the influence of $\mathbf{\Pi}_p$ over $\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$?

Spiked Models II

Remarks

- The spiked model is a particular case of **large covariance matrix** model with

$$\mathbf{R}_p = \mathbf{I}_p + \sum_{\ell=1}^{k} \theta_\ell \vec{\mathbf{u}}_\ell \vec{\mathbf{u}}_\ell^*$$

- There are additive spiked models: $\check{\mathbf{X}}_n = \mathbf{X}_n + \mathbf{A}_n$ where $\mathbf{A}_n$ is a matrix with finite rank.

Objective

- What is the influence of $\mathbf{\Pi}_p$ over $L_N\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$? None!
- What is the influence of $\mathbf{\Pi}_p$ over $\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$?

Spiked Models II

Remarks

- The spiked model is a particular case of **large covariance matrix** model with

$$\mathbf{R}_p = \mathbf{I}_p + \sum_{\ell=1}^{k} \theta_\ell \vec{\mathbf{u}}_\ell \vec{\mathbf{u}}_\ell^*$$

- There are additive spiked models: $\check{\mathbf{X}}_n = \mathbf{X}_n + \mathbf{A}_n$ where $\mathbf{A}_n$ is a matrix with finite rank.

Objective

- What is the influence of $\mathbf{\Pi}_p$ over $L_N\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$? None!
- What is the influence of $\mathbf{\Pi}_p$ over $\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$? Well, it depends!

Simulations I: Single spikes

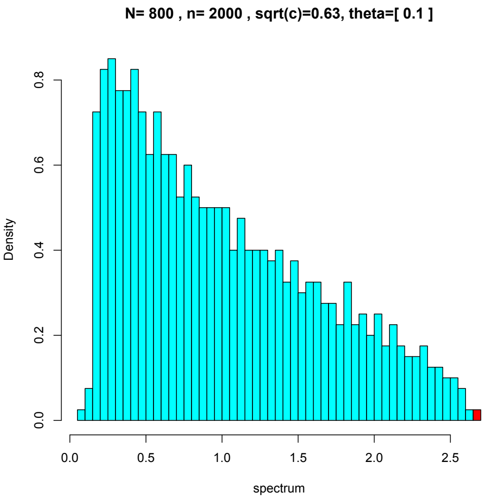

Figure: Spiked model - strength of the perturbation $\theta = 0.1$

Simulations I: Single spikes

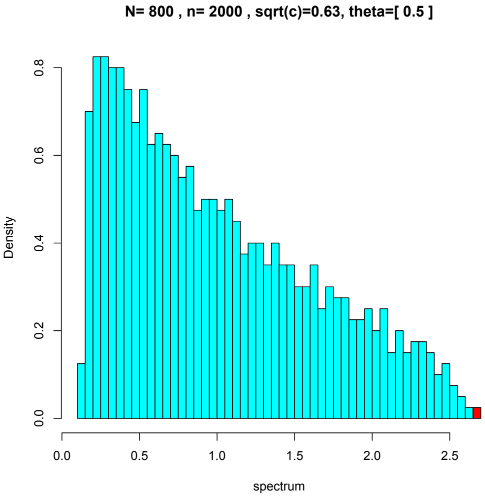

Figure: Spiked model - strength of the perturbation $\theta = 0.5$

Simulations I: Single spikes

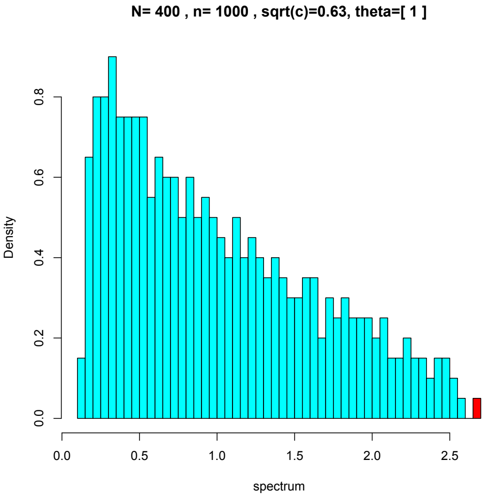

Figure: Spiked model - strength of the perturbation $\theta = 1$

Simulations I: Single spikes

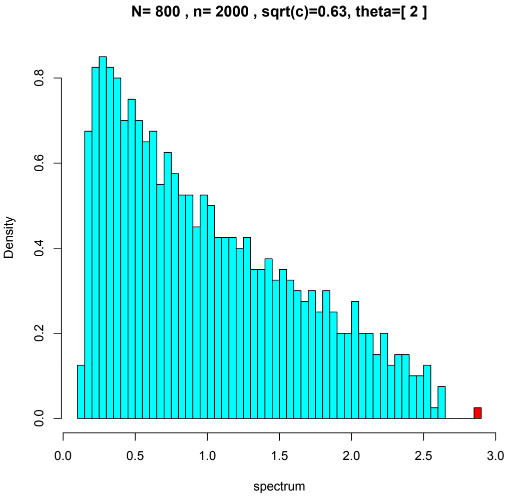

Figure: Spiked model - strength of the perturbation $\theta = 2$

Simulations I: Single spikes

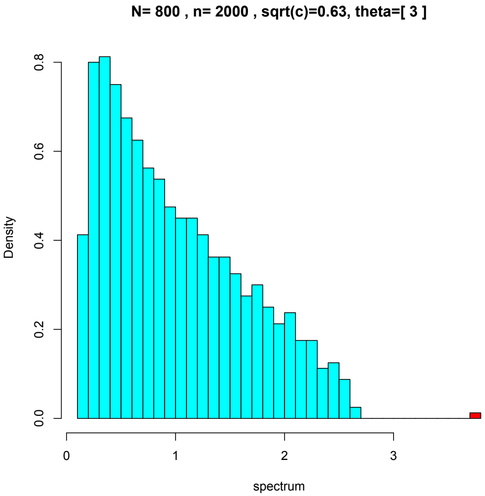

Figure: Spiked model - strength of the perturbation $\theta = 3$

Observation #1

If the **strength** θ of the perturbation $\mathbf{P}_p$ is large enough, then the limit of $\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$ is **strictly larger** than the right edge of the bulk.

Simulations II: Spectral measure

Simulations II: Spectral measure

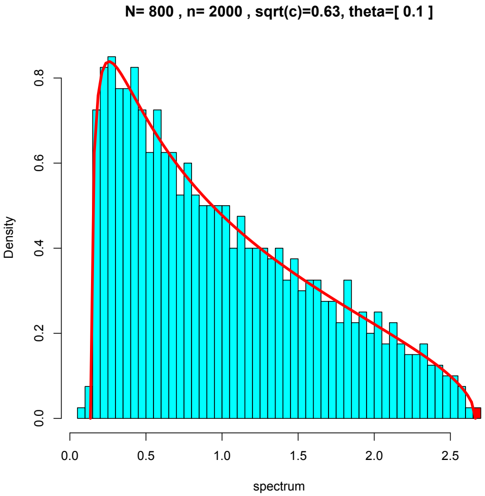

Figure: Spiked model - strength of the perturbation $\theta = 0.1$

Simulations II: Spectral measure

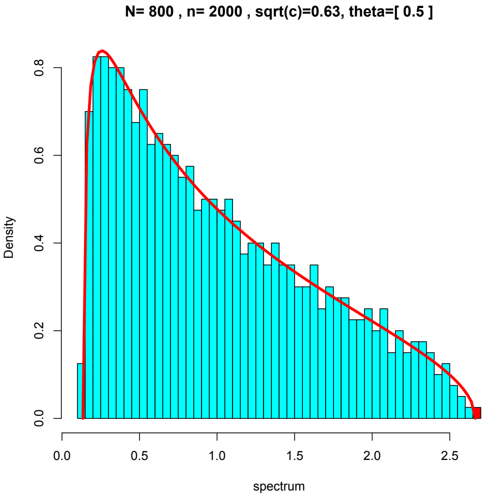

Figure: Spiked model - strength of the perturbation $\theta = 0.5$

Simulations II: Spectral measure

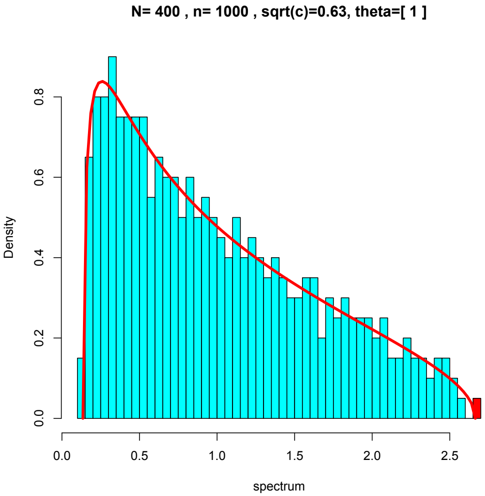

Figure: Spiked model - strength of the perturbation $\theta = 1$

Simulations II: Spectral measure

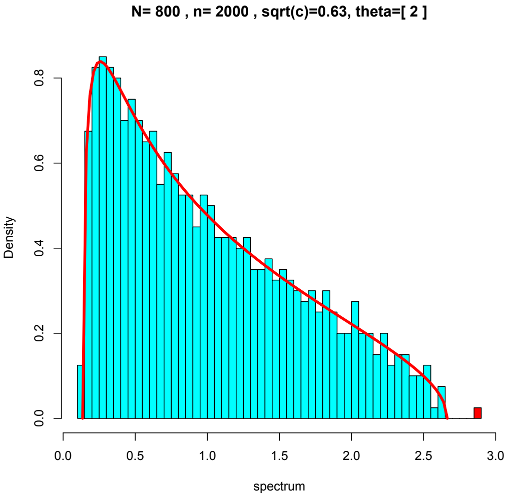

Figure: Spiked model - strength of the perturbation $\theta = 2$

Simulations II: Spectral measure

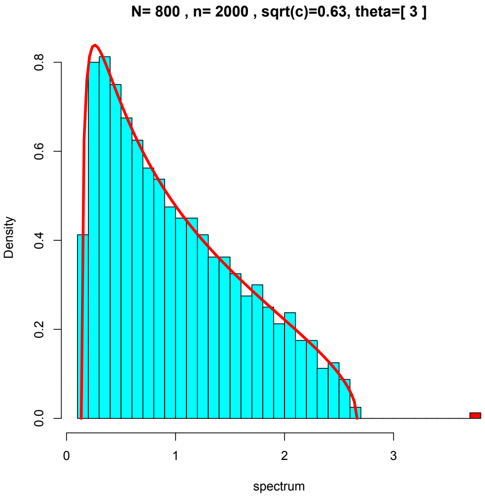

Figure: Spiked model - strength of the perturbation $\theta = 3$

Simulations III: Multiple Spikes

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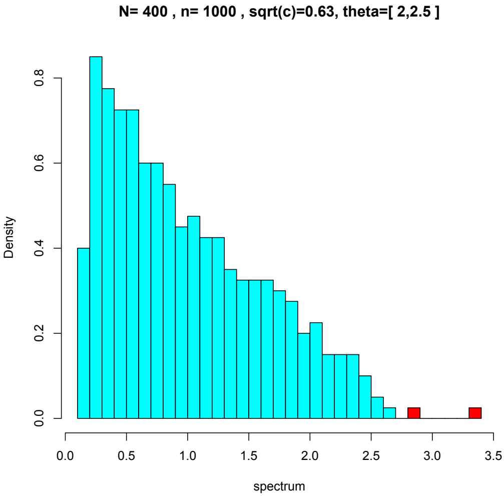


Figure: Spiked model - Two spikes

Simulations III: Multiple Spikes

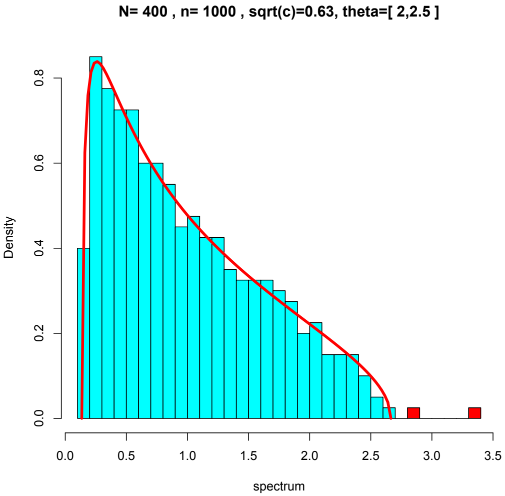


Figure: Spiked model - Two spikes

Simulations III: Multiple Spikes

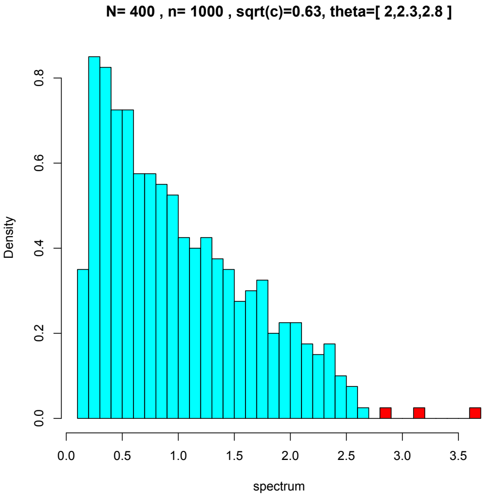


Figure: Spiked model - Three spikes

Simulations III: Multiple Spikes

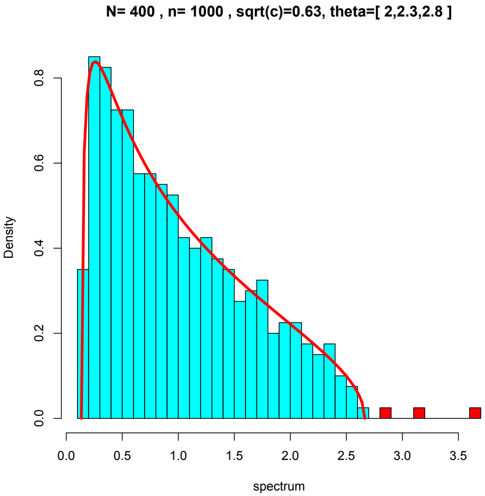


Figure: Spiked model - Three spikes

Simulations III: Multiple Spikes

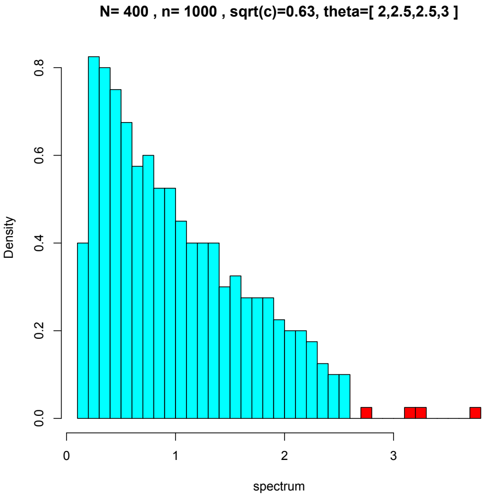


Figure: Spiked model - Multiple spikes

Simulations III: Multiple Spikes

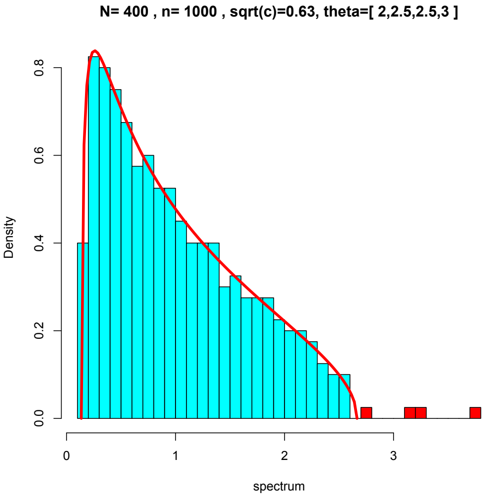


Figure: Spiked model - Multiple spikes

Observation # 2

Whathever the perturbations, the spectral measure converges toward Marčenko-Pastur distribution

The limiting spectral measure I

Theorem

The following convergence holds true: $\boxed{L_n\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right) \xrightarrow[p,n\to\infty]{a.s.} \mathbb{P}_{\check{\mathrm{MP}}}\ .}$

Remark

The limiting spectral measure is not sensitive to the presence of spikes

The limiting spectral measure II

Proof

The spiked model is a particular case of **large covariance matrix** model with

$$\mathbf{R}_p = \mathbf{I}_p + \sum_{\ell=1}^{k} \theta_\ell \vec{\mathbf{u}}_\ell \vec{\mathbf{u}}_\ell^*$$

Consider the spectral measure of $\mathbf{R}_p$ (orthogonal eigenvectors for the perturbations assumed):

$$L_n^{\mathbf{R}} = \frac{1}{p}\sum_{i=1}^{k}\delta_{1+\theta_i} + \frac{1}{p}\sum_{i=k+1}^{p}\delta_1 \xrightarrow[p,n\to\infty]{} \mathbb{P}^{\mathbf{R}} = \delta_1$$

hence the limiting canonical equation

$$\begin{aligned} \mathbf{t}(z) &= \int \frac{\mathbb{P}^{\mathbf{R}}(d\,\lambda)}{(1-c)\lambda - z - zc\mathbf{t}(z)\lambda} = \frac{1}{(1-c) - z - zc\mathbf{t}(z)} \\ \Leftrightarrow &\quad \boxed{zc\mathbf{t}^2 + [z-(1-c)]\mathbf{t} + 1 = 0} \end{aligned}$$

$\Rightarrow$ We recognize **Marčenko-Pastur canonical equation**.

Behaviour of the largest eigenvalue

We consider the following spiked model:

$$\widetilde{\mathbf{X}}_n = (\mathbf{I}_p + \theta \vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{1/2}\, \mathbf{X}_n \quad \text{with} \quad \|\vec{\mathbf{u}}\| = 1 \ .$$

which corresponds to a **rank-one** perturbation.

Theorem

Recall that $c = \lim_{n\to\infty} \frac{p}{n}$.

- if $\boxed{\theta \leq \sqrt{c}}$ then

$$\lambda_{\max} = \lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right) \xrightarrow[p,n\to\infty]{a.s.} \sigma^2(1+\sqrt{c})^2$$

- if $\boxed{\theta > \sqrt{c}}$ then

$$\lambda_{\max} \xrightarrow[p,n\to\infty]{a.s.} \sigma^2(1+\theta)\left(1+\frac{c}{\theta}\right) > \sigma^2(1+\sqrt{c})^2$$

Phase transition Phenomenon

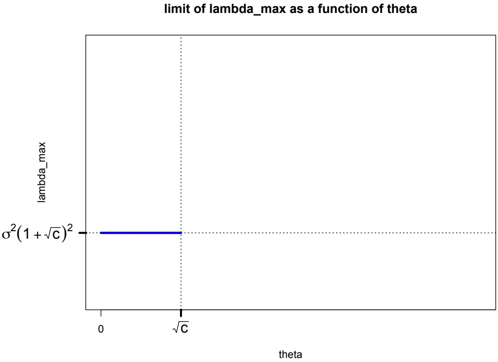


Figure: Limit of largest eigenvalue $\lambda_{\max}$ as a function of the perturbation θ

Phase transition Phenomenon

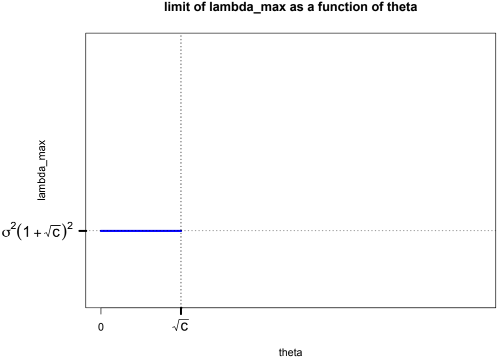


Figure: Limit of largest eigenvalue $\lambda_{\max}$ as a function of the perturbation θ

- If $\theta \leq \sqrt{c}$ then

$$\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right) \xrightarrow[p,n\to\infty]{} \sigma^2(1+\sqrt{c})^2 \ .$$

Phase transition Phenomenon

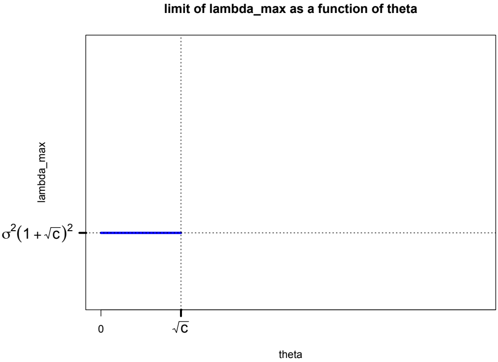


Figure: Limit of largest eigenvalue $\lambda_{\max}$ as a function of the perturbation θ

- If $\theta \leq \sqrt{c}$ then

$$\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right) \xrightarrow[p,n\to\infty]{} \sigma^2(1+\sqrt{c})^2 \ .$$

$$\boxed{\text{Below the threshold } \sqrt{c},\ \lambda_{\max}\left(\tfrac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right) \text{ asymptotically } \textbf{sticks to the bulk}.}$$

Phase transition Phenomenon

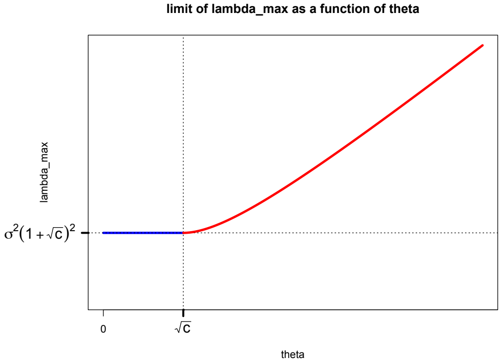


Figure: Limit of largest eigenvalue $\lambda_{\max}$ as a function of the perturbation θ

Phase transition Phenomenon

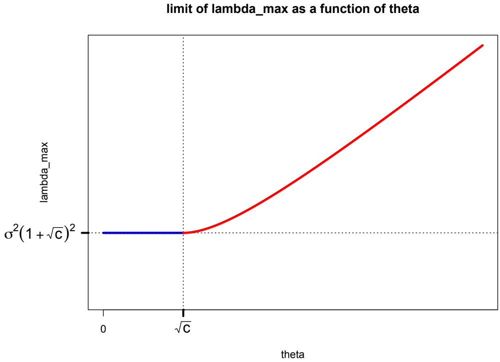


Figure: Limit of largest eigenvalue $\lambda_{\max}$ as a function of the perturbation θ

- if $\theta > \sqrt{c}$ then

$$\lim_{p,n\to\infty} \lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right) = \sigma^2(1+\theta)\left(1+\frac{c}{\theta}\right)$$

Phase transition Phenomenon

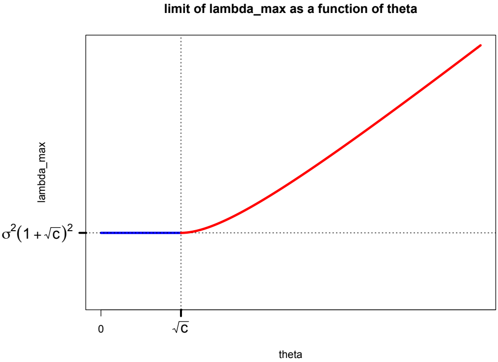


Figure: Limit of largest eigenvalue $\lambda_{\max}$ as a function of the perturbation θ

- if $\theta > \sqrt{c}$ then

$$\lim_{p,n\to\infty} \lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right) \quad = \quad \sigma^2(1+\theta)\left(1+\frac{c}{\theta}\right) > \sigma^2\left(1+\sqrt{c}\right)^2$$

Above the threshold $\sqrt{c}$, $\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$ asymptotically **separates from the bulk**.

Strategy of proof

1. We first express a condition for which

$$\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$$

separates from the bulk and refer to it as $\boxed{\textbf{the determinant condition}}$

2. Relying on Large Random Matrix theory, we simplify this condition and obtain

$$\boxed{\textbf{the asymptotic condition}}$$

3. We finally conclude, obtain the condition $\boxed{\theta > \sqrt{c}}$ for which the limit of $\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$ separates from the bulk, and compute this limit.

Notations

- Marčenko-Pastur model

$$\mathbf{Z}_n = \frac{1}{n}\mathbf{X}_n\mathbf{X}_n^* \qquad \text{and} \qquad \mathbf{Q}(z) = (-z\mathbf{I}_p + \mathbf{Z}_n)^{-1}$$

- Spiked model

$$\widetilde{\mathbf{X}}_n = \mathbf{\Pi}^{1/2}\mathbf{X}_n = (\mathbf{I}_p + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{1/2}\,\mathbf{X}_n \qquad \text{and} \qquad \widetilde{\mathbf{Z}}_n = \frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*$$

The determinant condition I

We wish to find

- λ^θ eigenvalue of the spiked model

$$\widetilde{\mathbf{Z}}_n = \frac{1}{n}\mathbf{\Pi}^{1/2}\mathbf{X}_n\mathbf{X}_n^*\mathbf{\Pi}^{1/2}$$

- λ^θ **not** an eigenvalue of M̌P model

$$\mathbf{Z}_n = \frac{1}{n}\mathbf{X}_n\mathbf{X}_n^*$$

Otherwise stated

$$\boxed{\det\left(-\lambda^\theta \mathbf{I} + \widetilde{\mathbf{Z}}\right) = 0} \quad \textbf{but} \quad \boxed{\det\left(-\lambda^\theta \mathbf{I} + \mathbf{Z}\right) \neq 0}$$

Inverse of a rank-one perturbation of the identity

Recall that

$$\mathbf{\Pi}_N = \mathbf{I} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*$$

Standard results from linear algebra yield

$$\mathbf{\Pi}_N^{-1} = (\mathbf{I} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1} = \mathbf{I} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*$$

The determinant condition II

Let's go for **simple** computations:

$$\det\left(-\lambda^{\theta}\mathbf{I}+\widetilde{\mathbf{Z}}\right)=0 \quad \Leftrightarrow \quad \det\left(-\lambda^{\theta}\mathbf{I}+\mathbf{\Pi}_N^{1/2}\mathbf{Z}\mathbf{\Pi}_N^{1/2}\right)=0$$

The determinant condition II

Let's go for **simple** computations:

$$\det\left(-\lambda^\theta \mathbf{I} + \widetilde{\mathbf{Z}}\right) = 0 \quad \Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{I} + \mathbf{\Pi}_N^{1/2}\mathbf{Z}\mathbf{\Pi}_N^{1/2}\right) = 0$$

$$\Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{\Pi}_N^{-1} + \mathbf{Z}\right) = 0$$

The determinant condition II

Let's go for **simple** computations:

$$\det\left(-\lambda^\theta \mathbf{I} + \widetilde{\mathbf{Z}}\right) = 0 \quad \Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{I} + \mathbf{\Pi}_N^{1/2}\mathbf{Z}\mathbf{\Pi}_N^{1/2}\right) = 0$$

$$\Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{\Pi}_N^{-1} + \mathbf{Z}\right) = 0$$

$$\Leftrightarrow \quad \det\left(-\lambda^\theta \left(\mathbf{I} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right) + \mathbf{Z}\right) = 0$$

The determinant condition II

Let's go for **simple** computations:

$$\det\left(-\lambda^\theta \mathbf{I} + \widetilde{\mathbf{Z}}\right) = 0 \quad \Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{I} + \mathbf{\Pi}_N^{1/2}\mathbf{Z}\mathbf{\Pi}_N^{1/2}\right) = 0$$

$$\Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{\Pi}_N^{-1} + \mathbf{Z}\right) = 0$$

$$\Leftrightarrow \quad \det\left(-\lambda^\theta\left(\mathbf{I}_N - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right) + \mathbf{Z}\right) = 0$$

$$\Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{I} + \mathbf{Z} + \lambda^\theta \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right) = 0$$

The determinant condition II

Let's go for **simple** computations:

$$
\begin{aligned}
\det\left(-\lambda^\theta \mathbf{I} + \widetilde{\mathbf{Z}}\right) = 0 \quad &\Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{I} + \mathbf{\Pi}_N^{1/2}\mathbf{Z}\mathbf{\Pi}_N^{1/2}\right) = 0 \\
&\Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{\Pi}_N^{-1} + \mathbf{Z}\right) = 0 \\
&\Leftrightarrow \quad \det\left(-\lambda^\theta \left(\mathbf{I} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right) + \mathbf{Z}\right) = 0 \\
&\Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{I} + \mathbf{Z} + \lambda^\theta \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right) = 0 \\
&\Leftrightarrow \quad \det\left[\left(-\lambda^\theta \mathbf{I} + \mathbf{Z}\right)\left(\mathbf{I} + \lambda^\theta \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^* \mathbf{Q}(\lambda^\theta)\right)\right] = 0
\end{aligned}
$$

The determinant condition II

Let's go for **simple** computations:

$$
\begin{aligned}
\det\left(-\lambda^\theta \mathbf{I} + \widetilde{\mathbf{Z}}\right) = 0 \quad &\Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{I} + \mathbf{\Pi}_N^{1/2}\mathbf{Z}\mathbf{\Pi}_N^{1/2}\right) = 0 \\
&\Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{\Pi}_N^{-1} + \mathbf{Z}\right) = 0 \\
&\Leftrightarrow \quad \det\left(-\lambda^\theta \left(\mathbf{I} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right) + \mathbf{Z}\right) = 0 \\
&\Leftrightarrow \quad \det\left(-\lambda^\theta \mathbf{I} + \mathbf{Z} + \lambda^\theta \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right) = 0 \\
&\Leftrightarrow \quad \det\left[\left(-\lambda^\theta \mathbf{I} + \mathbf{Z}\right)\left(\mathbf{I} + \lambda^\theta \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}(\lambda^\theta)\right)\right] = 0 \\
&\Leftrightarrow \quad \boxed{\det\left[\mathbf{I} + \lambda^\theta \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}(\lambda^\theta)\right] = 0}
\end{aligned}
$$

The determinant condition II

Let's go for **simple** computations:

$$
\begin{aligned}
\det\left(-\lambda^{\theta}\mathbf{I}+\widetilde{\mathbf{Z}}\right)=0 \quad &\Leftrightarrow \quad \det\left(-\lambda^{\theta}\mathbf{I}+\mathbf{\Pi}_N^{1/2}\mathbf{Z}\mathbf{\Pi}_N^{1/2}\right)=0\\
&\Leftrightarrow \quad \det\left(-\lambda^{\theta}\mathbf{\Pi}_N^{-1}+\mathbf{Z}\right)=0\\
&\Leftrightarrow \quad \det\left(-\lambda^{\theta}\left(\mathbf{I}-\frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)+\mathbf{Z}\right)=0\\
&\Leftrightarrow \quad \det\left(-\lambda^{\theta}\mathbf{I}+\mathbf{Z}+\lambda^{\theta}\frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)=0\\
&\Leftrightarrow \quad \det\left[\left(-\lambda^{\theta}\mathbf{I}+\mathbf{Z}\right)\left(\mathbf{I}+\lambda^{\theta}\frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\mathbf{Q}(\lambda^{\theta})\right)\right]=0\\
&\Leftrightarrow \quad \boxed{\det\left[\mathbf{I}+\lambda^{\theta}\frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\mathbf{Q}(\lambda^{\theta})\right]=0}
\end{aligned}
$$

Interest of this expression

In this equation, perturbation features θ and $\vec{\mathbf{u}}$ are separated from the resolvent of $\check{\text{M}}$P model (non-spiked model)

The determinant condition III

A useful formula

Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. Recall the formula

$$\boxed{\det(I_m + AB) = \det(I_n + BA)}.$$

For a one-line proof of this property, write

$$M = \begin{pmatrix} I_m & -A \\ B & I_n \end{pmatrix} = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} \begin{pmatrix} I & -A \\ 0 & I + BA \end{pmatrix} = \begin{pmatrix} I & -A \\ 0 & I \end{pmatrix} \begin{pmatrix} I + AB & 0 \\ B & I \end{pmatrix}$$

And compute the determinant of M following each decomposition.

New formulation of the condition

The **determinant condition** writes

$$\det\left[\mathbf{I} + \lambda^\theta \frac{\theta}{1+\theta} \vec{\mathbf{u}}\vec{\mathbf{u}}^* \mathbf{Q}(\lambda^\theta)\right] = 0 \quad \Leftrightarrow \quad \lambda^\theta \frac{\theta}{1+\theta} \vec{\mathbf{u}}^* \mathbf{Q}(\lambda^\theta)\vec{\mathbf{u}} = -1$$

$$\Leftrightarrow \quad \boxed{\lambda^\theta\, \vec{\mathbf{u}}^* \mathbf{Q}(\lambda^\theta)\vec{\mathbf{u}} = -\frac{1+\theta}{\theta}}$$

The asymptotic condition I

Recall the condition

$$\boxed{\lambda^{\theta}\vec{\mathbf{u}}^{*}\mathbf{Q}(\lambda^{\theta})\vec{\mathbf{u}} = -\frac{1+\theta}{\theta}}$$

Asymptotic simplification

By **Isotropic MP theorem**, we have

$$\vec{\mathbf{u}}^{*}\mathbf{Q}(\lambda^{\theta})\vec{\mathbf{u}} \xrightarrow[p,n\to\infty]{} \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}\left(\lambda^{\theta}\right) .$$

Hence the final form of the condition

$$\boxed{\lambda^{\theta}\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}\left(\lambda^{\theta}\right) = -\frac{1+\theta}{\theta}}$$

The asymptotic condition II

- We introduce the following function $\rho(z)$:

$$\boxed{\rho(z) = 1 + z\, g(z)}$$

- Let $\rho_{\check{\mathrm{M}}\mathrm{P}}$ associated to the Stieltjes transform:

$$\rho_{\check{\mathrm{M}}\mathrm{P}}(z) = 1 + z\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)\ .$$

  Then the condition over λ^θ writes:

$$\lambda^\theta \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}\left(\lambda^\theta\right) = -\frac{1+\theta}{\theta} \quad \Leftrightarrow \quad \rho_{\check{\mathrm{M}}\mathrm{P}}(\lambda^\theta) - 1 = -\frac{1+\theta}{\theta}$$

$$\Leftrightarrow \quad \boxed{\rho_{\check{\mathrm{M}}\mathrm{P}}\left(\lambda^\theta\right) = -\frac{1}{\theta}}$$

The asymptotic condition III

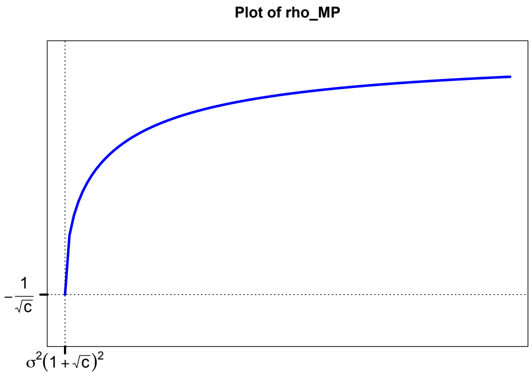


Figure: Plot of $\rho_{\check{\mathrm{M}}\mathrm{P}}$ on $(\sigma^2(1+\sqrt{c})^2, \infty)$

The function $\rho_{\check{\mathrm{M}}\mathrm{P}}$ admits an explicit expression on $(\sigma^2(1+\sqrt{c})^2, \infty)$

$$\rho_{\check{\mathrm{M}}\mathrm{P}}(x) = 1 + \frac{1}{2c}\left\{(1-x-c) + \sqrt{(1-x-c)^2 - 4cx}\right\} \quad (\sigma^2 = 1)$$

The asymptotic condition III

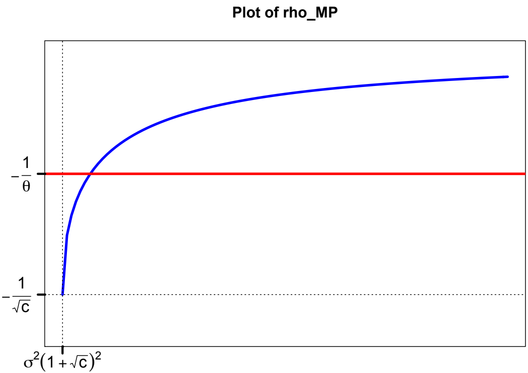


Figure: Plot of $\rho_{\check{\mathrm{M}}\mathrm{P}}$ on $(\sigma^2(1+\sqrt{c})^2, \infty)$

The asymptotic condition is satisfied if

$$\rho_{\check{\mathrm{M}}\mathrm{P}}\left(\lambda^\theta\right) = -\frac{1}{\theta} \quad \Leftrightarrow \quad -\frac{1}{\theta} > -\frac{1}{\sqrt{c}} \quad \Leftrightarrow \quad \boxed{\theta > \sqrt{c}}$$

Computing the limit λ^θ

We have

$$\rho_{\check{\mathrm{M}}\mathrm{P}}\left(\lambda^\theta\right) = -\frac{1}{\theta} \quad \Leftrightarrow \quad \boxed{\lambda^\theta = \rho_{\check{\mathrm{M}}\mathrm{P}}^{-1}\left(-\frac{1}{\theta}\right)}$$

We therefore need to inverse $\rho_{\check{\mathrm{M}}\mathrm{P}}$.

- Using Marčenko-Pastur equation and the relation between $\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}$ and $\rho_{\check{\mathrm{M}}\mathrm{P}}$

$$\begin{aligned}
\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z) &= \frac{1}{\sigma^2(1-c) - z - z\sigma^2 c\,\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)} \\
\rho_{\check{\mathrm{M}}\mathrm{P}}(z) &= 1 + z\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)
\end{aligned}$$

  we get

$$z = \frac{\sigma^2}{\rho_{\check{\mathrm{M}}\mathrm{P}}(z)}\left(\rho_{\check{\mathrm{M}}\mathrm{P}}(z) - 1\right)\left(1 - c\rho_{\check{\mathrm{M}}\mathrm{P}}(z)\right)$$

- Replacing now $z = \rho_{\check{\mathrm{M}}\mathrm{P}}^{-1}\left(-\frac{1}{\theta}\right)$ into the equation yields:

$$\boxed{\lambda^\theta = \rho_{\check{\mathrm{M}}\mathrm{P}}^{-1}\left(-\frac{1}{\theta}\right) = \sigma^2(1+\theta)\left(1 + \frac{c}{\theta}\right)}$$

Spiked model eigenvectors I

- Consider the following $p \times n$ spiked model:

$$\begin{aligned} \widetilde{\mathbf{X}}_n &= (\mathbf{I} + \theta \vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{1/2}\,\mathbf{X}_n \quad \text{with} \quad \|\vec{\mathbf{u}}\| = 1\ , \\ &= \mathbf{\Pi}^{1/2}\mathbf{X}_n \end{aligned}$$

  where $\mathbf{X}_n$ has i.i.d. $0/\sigma^2$ entries.
- Let $\vec{\boldsymbol{v}}_{\max}$ be the eigenvector associated to $\lambda_{\max}$, the largest eigenvalue of the covariance matrix associated to $\widetilde{\mathbf{X}}_n$:

$$\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)\vec{\boldsymbol{v}}_{\max} = \lambda_{\max}\vec{\boldsymbol{v}}_{\max}$$

Question

- What is the behavior of $\vec{\boldsymbol{v}}_{\max}$ as $p, n \to \infty$ in the regime where

$$\frac{p}{n} \to c \in (0, \infty)?$$

Reminder

Behaviour of largest eigenvalue $\lambda_{\max}$ well-understood:

- if $\boxed{\theta \leq \sqrt{c}}$ then $\lambda_{\max} \to \sigma^2(1+\sqrt{c})^2$, **the right edge of MP bulk.**
- if $\boxed{\theta > \sqrt{c}}$ then $\lambda_{\max} \to \sigma^2(1+\theta)(1+c/\theta)$, i.e. $\lambda_{\max}$ **separates from the bulk.**

Spiked model eigenvectors II

Preliminary observations

1. Let p finite, $n \to \infty$, then

$$\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^* = \mathbf{\Pi}^{1/2}\left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n^*\right)\mathbf{\Pi}^{1/2} \xrightarrow[n\to\infty]{} \mathbf{\Pi}$$

Largest eigenvalue of $\mathbf{\Pi}$ is $1+\theta$; associated eigenvector is $\vec{\mathbf{u}}$:

$$\mathbf{\Pi}\vec{\mathbf{u}} = (\mathbf{I}_p + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)\,\vec{\mathbf{u}} = (1+\theta)\vec{\mathbf{u}}\ .$$

As a consequence:

$$\vec{\boldsymbol{v}}_{\max} \xrightarrow[n\to\infty]{} \vec{\mathbf{u}}\ .$$

2. If

$$p, n \to \infty\ , \qquad \frac{p}{n} \to c\ ,$$

then $\boxed{\dim(\vec{\boldsymbol{v}}_{\max}) = p \nearrow \infty}$. We therefore consider the projection

$$\vec{\boldsymbol{v}}_{\max}\vec{\boldsymbol{v}}_{\max}^*$$

of $\vec{\boldsymbol{v}}_{\max}$ on a generic deterministic vector $\vec{\boldsymbol{a}}_N \in \mathbb{R}^p$, i.e.

$$\boxed{\vec{\boldsymbol{a}}_N^*\vec{\boldsymbol{v}}_{\max}\vec{\boldsymbol{v}}_{\max}^*\vec{\boldsymbol{a}}_N} \quad = \quad |\langle\vec{\boldsymbol{v}}_{\max}, \vec{\boldsymbol{a}}_N\rangle|^2$$

Spiked model eigenvectors III

Theorem

Let $(\vec{\boldsymbol{a}}_N)$ be a family of deterministic vectors with norm 1, then

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N - \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1} \vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N \xrightarrow[p,n\to\infty]{a.s.} 0 \ .$$

Remarks

- If p finite, $n \to \infty$, then

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N - \vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N \xrightarrow[p,n\to\infty]{a.s.} 0 \ .$$

- The large dimension $\frac{p}{n} \to c$ induces a correction factor:

$$\boxed{\kappa(c) = \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1}}$$

- Of course $\kappa(c) \to 1$ if $c \to 0$.

Outline of proof

1. Expression of $\vec{\boldsymbol{v}}_{\max}$ with the help of the resolvent

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N = \frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \widetilde{Q}(z) \vec{\boldsymbol{a}}_N \, dz$$

2. Convenient expression of $\vec{\boldsymbol{v}}_{\max}$ where the contribution of the perturbation is separated from the resolvent of the non-perturbated model ($\check{\mathrm{M}}$P)

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N \quad \approx \quad -\frac{\vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)} \, dz$$

3. Residue calculus to find the final form

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N - \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1} \vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N \xrightarrow[p,n \to \infty]{a.s.} 0 \, .$$

Proof I

Reminder from complex analysis

We need a simple result from complex analysis:

$$\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}-} \frac{dz}{z} = 1$$

if $\mathcal{C}^-$ is a contour (take a circle of radius 1) enclosing counterclockwise 0.

- Proof:

$$\text{let } z = e^{i\theta} : \qquad \frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}-} \frac{dz}{z} = \frac{1}{2\boldsymbol{i}\pi} \int_0^{2\pi} \frac{d(e^{\boldsymbol{i}\theta})}{e^{\boldsymbol{i}\theta}} = \frac{1}{2\boldsymbol{i}\pi} \int_0^{2\pi} \frac{\boldsymbol{i}e^{\boldsymbol{i}\theta} d\theta}{e^{\boldsymbol{i}\theta}} = 1$$

In particular, if $\mathcal{C}^+$ is a contour enclosing λ **clockwise**, then:

$$\boxed{\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \frac{dz}{\lambda - z} = 1}$$

(let $\mathcal{C}^+$ be a circle $(\lambda + \rho e^{\boldsymbol{i}\theta}; 0 \leq \theta \leq 2\pi)$ and perform a change of variable).

If $\mathcal{C}^+$ does not enclose λ, then the integral **equals zero**.

Proof II

Our objective

$$\boxed{\text{To express } \vec{\boldsymbol{v}}_{\max} \text{ with the help of the } \textbf{resolvent } \widetilde{\mathbf{Q}}(z) = \left(\tfrac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^* - z\mathbf{I}_p\right)^{-1}}$$

By the spectral theorem,

$$\begin{aligned}
\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^* &= \boldsymbol{O}_p \begin{pmatrix} \lambda_{\max} & & \\ & \ddots & \\ & & \lambda_p \end{pmatrix} \boldsymbol{O}_p^* \\
&= [\vec{\boldsymbol{v}}_{\max}\ \boldsymbol{O}_{p-1}] \begin{pmatrix} \lambda_{\max} & & \\ & \ddots & \\ & & \lambda_p \end{pmatrix} \begin{bmatrix} \vec{\boldsymbol{v}}_{\max}^* \\ \boldsymbol{O}_{p-1}^* \end{bmatrix}
\end{aligned}$$

In particular,

$$\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^* - z\mathbf{I}_p\right)^{-1} = [\vec{\boldsymbol{v}}_{\max}\ \boldsymbol{O}_{p-1}] \begin{pmatrix} \frac{1}{\lambda_{\max}-z} & & \\ & \ddots & \\ & & \frac{1}{\lambda_p-z} \end{pmatrix} \begin{bmatrix} \vec{\boldsymbol{v}}_{\max}^* \\ \boldsymbol{O}_{p-1}^* \end{bmatrix}$$

Recall that

- if $\theta > \sqrt{c}$, $\lambda_{\max}$ separates from the bulk

and consider a contour $\mathcal{C}^+$ **exclusively enclosing** the eigenvalue $\lambda_{\max}$.

Proof III

We have

$$\boxed{\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N = \frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \widetilde{Q}(z) \vec{\boldsymbol{a}}_N \, dz}$$

Indeed,

$$\begin{aligned}
&\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \widetilde{Q}(z) \vec{\boldsymbol{a}}_N \, dz \\
&= \frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* [\vec{\boldsymbol{v}}_{\max} \ \boldsymbol{O}_{p-1}] \begin{pmatrix} \frac{1}{\lambda_{\max}-z} & & \\ & \ddots & \\ & & \frac{1}{\lambda_p - z} \end{pmatrix} \begin{bmatrix} \vec{\boldsymbol{v}}_{\max}^* \\ \boldsymbol{O}_{p-1}^* \end{bmatrix} \vec{\boldsymbol{a}}_N \, dz \\
&= \vec{\boldsymbol{a}}_N^* [\vec{\boldsymbol{v}}_{\max} \ \boldsymbol{O}_{p-1}] \begin{pmatrix} \frac{1}{2\boldsymbol{i}\pi} \oint \frac{1}{\lambda_{\max}-z} dz & & \\ & \ddots & \\ & & \frac{1}{2\boldsymbol{i}\pi} \oint \frac{1}{\lambda_p - z} dz \end{pmatrix} \begin{bmatrix} \vec{\boldsymbol{v}}_{\max}^* \\ \boldsymbol{O}_{p-1}^* \end{bmatrix} \vec{\boldsymbol{a}}_N \\
&= \vec{\boldsymbol{a}}_N^* [\vec{\boldsymbol{v}}_{\max} \ \boldsymbol{O}_{p-1}] \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \begin{bmatrix} \vec{\boldsymbol{v}}_{\max}^* \\ \boldsymbol{O}_{p-1}^* \end{bmatrix} \vec{\boldsymbol{a}}_N \\
&= \vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N \ .
\end{aligned}$$

Proof IV

Recall

$$\frac{1}{2\boldsymbol{i}\pi}\oint_{\mathcal{C}+}\vec{\boldsymbol{a}}_N^*\widetilde{\mathbf{Q}}(z)\vec{\boldsymbol{a}}_N\,dz$$

and temporarily forget about the integral. Our objective now is:

to find a new formulation of $\vec{\boldsymbol{a}}_N^*\widetilde{\mathbf{Q}}(\mathbf{z})\vec{\boldsymbol{a}}_{\mathbf{N}}$ and clearly separate the contribution from the perturbation ($\vec{\mathbf{u}}$ and θ) and the resolvent $\mathbf{Q}(z)$ from the non-pertubated model.

Introduce the notations

$$\mathbf{Z} = \frac{1}{n}\mathbf{X}_n\mathbf{X}_n^* \quad \text{and} \quad \widetilde{\mathbf{Z}} = \frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*$$

and recall the formula for the inverse of a rank-one perturbation:

$$(\mathbf{A}+\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{A}^{-1}}{1+\vec{\mathbf{u}}\mathbf{A}\vec{\mathbf{u}}^*},$$

In particular

$$\mathbf{\Pi}^{-1} = (\mathbf{I}_p+\theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1} = \mathbf{I}_p - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*$$

Proof V

$$\widetilde{\mathbf{Q}}(z) = \left(\mathbf{\Pi}^{1/2}\mathbf{Z}\mathbf{\Pi}^{1/2} - z\mathbf{I}\right)^{-1}$$

Proof V

$$\begin{aligned}\widetilde{\mathbf{Q}}(z) &= \left(\mathbf{\Pi}^{1/2}\mathbf{Z}\mathbf{\Pi}^{1/2} - z\mathbf{I}\right)^{-1} \\ &= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\mathbf{\Pi}^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2}\end{aligned}$$

Proof V

$$
\begin{aligned}
\widetilde{\mathbf{Q}}(z) &= \left(\mathbf{\Pi}^{1/2}\mathbf{Z}\mathbf{\Pi}^{1/2} - z\mathbf{I}\right)^{-1} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\mathbf{\Pi}^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z(\mathbf{I} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2}
\end{aligned}
$$

Proof V

$$\begin{aligned}
\widetilde{\mathbf{Q}}(z) &= \left(\mathbf{\Pi}^{1/2}\mathbf{Z}\mathbf{\Pi}^{1/2} - z\mathbf{I}\right)^{-1} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\mathbf{\Pi}^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z(\mathbf{I} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\left(\mathbf{I} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right)\right)^{-1}\mathbf{\Pi}^{-1/2}
\end{aligned}$$

Proof V

$$
\begin{aligned}
\widetilde{\mathbf{Q}}(z) &= \left(\mathbf{\Pi}^{1/2}\mathbf{Z}\mathbf{\Pi}^{1/2} - z\mathbf{I}\right)^{-1} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\mathbf{\Pi}^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z(\mathbf{I} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\left(\mathbf{I} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right)\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}(\mathbf{Z} - z\mathbf{I} + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\mathbf{\Pi}^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta}
\end{aligned}
$$

Proof V

$$
\begin{aligned}
\widetilde{\mathbf{Q}}(z) &= \left(\mathbf{\Pi}^{1/2}\mathbf{Z}\mathbf{\Pi}^{1/2} - z\mathbf{I}\right)^{-1} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\mathbf{\Pi}^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z(\mathbf{I} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\left(\mathbf{I} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right)\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}(\mathbf{Z} - z\mathbf{I} + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\mathbf{\Pi}^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Q} - \frac{\mathbf{Q}\xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}\vec{\mathbf{u}}}\right)\mathbf{\Pi}^{-1/2}
\end{aligned}
$$

Proof V

$$
\begin{aligned}
\widetilde{\mathbf{Q}}(z) &= \left(\mathbf{\Pi}^{1/2}\mathbf{Z}\mathbf{\Pi}^{1/2} - z\mathbf{I}\right)^{-1} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\mathbf{\Pi}^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z(\mathbf{I} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\left(\mathbf{I} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right)\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}(\mathbf{Z} - z\mathbf{I} + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\mathbf{\Pi}^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Q} - \frac{\mathbf{Q}\xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}\vec{\mathbf{u}}}\right)\mathbf{\Pi}^{-1/2}
\end{aligned}
$$

Hence

$$
\vec{\boldsymbol{a}}_N^*\widetilde{\mathbf{Q}}(z)\vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\mathbf{Q}(z)\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N - \xi\frac{\vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\mathbf{Q}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}\vec{\mathbf{u}}}
$$

Proof V

$$\begin{aligned}
\widetilde{\mathbf{Q}}(z) &= \left(\mathbf{\Pi}^{1/2}\mathbf{Z}\mathbf{\Pi}^{1/2} - z\mathbf{I}\right)^{-1} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\mathbf{\Pi}^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z(\mathbf{I} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\left(\mathbf{I} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right)\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}(\mathbf{Z} - z\mathbf{I} + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\mathbf{\Pi}^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Q} - \frac{\mathbf{Q}\xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}\vec{\mathbf{u}}}\right)\mathbf{\Pi}^{-1/2}
\end{aligned}$$

Hence

$$\vec{\boldsymbol{a}}_N^*\widetilde{\mathbf{Q}}(z)\vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\mathbf{Q}(z)\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N - \xi\frac{\vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\mathbf{Q}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}\vec{\mathbf{u}}}$$

Not so ugly!

Proof V

$$
\begin{aligned}
\widetilde{\mathbf{Q}}(z) &= \left(\mathbf{\Pi}^{1/2}\mathbf{Z}\mathbf{\Pi}^{1/2} - z\mathbf{I}\right)^{-1} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\mathbf{\Pi}^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z(\mathbf{I} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z} - z\left(\mathbf{I} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right)\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}(\mathbf{Z} - z\mathbf{I} + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\mathbf{\Pi}^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Q} - \frac{\mathbf{Q}\xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}\vec{\mathbf{u}}}\right)\mathbf{\Pi}^{-1/2}
\end{aligned}
$$

Hence

$$
\vec{\boldsymbol{a}}_N^*\widetilde{\mathbf{Q}}(z)\vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\mathbf{Q}(z)\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N - \xi\frac{\vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\mathbf{Q}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}\vec{\mathbf{u}}}
$$

Not so ugly! And we have separated the contribution of the perturbation from the non-perturbated model.

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \widetilde{\mathbf{Q}}(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q} \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q} \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q} \vec{\mathbf{u}}}$$

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \widetilde{\mathbf{Q}}(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2} \mathbf{Q}(z) \boldsymbol{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2} \mathbf{Q} \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q} \boldsymbol{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q} \vec{\mathbf{u}}}$$

and integrate the first term

$$\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2} \mathbf{Q}(z) \boldsymbol{\Pi}^{1/2} \vec{\boldsymbol{a}}_N = ??$$

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \widetilde{\mathbf{Q}}(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q} \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q} \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q} \vec{\mathbf{u}}}$$

and integrate the first term

$$\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N = \mathbf{0}$$

Why?

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \widetilde{\mathbf{Q}}(z)\vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2}\mathbf{Q}(z)\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2}\mathbf{Q}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}\vec{\mathbf{u}}}$$

and integrate the first term

$$\frac{1}{2\boldsymbol{i}\pi}\oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2}\mathbf{Q}(z)\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N = 0$$

Why? Because

1. the contour only encloses $\lambda_{\max}(\widetilde{\mathbf{Z}})$ which is away from the bulk,
2. but all the eigenvalues of $\mathbf{Z}$ are **in the bulk**. Hence:

$$\frac{1}{2\boldsymbol{i}\pi}\oint_{\mathcal{C}+} \frac{1}{\lambda_i(\mathbf{Z})-z}\,dz = 0.$$

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \widetilde{\mathbf{Q}}(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q} \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q} \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q} \vec{\mathbf{u}}}$$

and integrate the first term

$$\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N = 0$$

Why? Because

1. the contour only encloses $\lambda_{\max}(\widetilde{\mathbf{Z}})$ which is away from the bulk,
2. but all the eigenvalues of $\mathbf{Z}$ are **in the bulk**. Hence:

$$\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \frac{1}{\lambda_i(\mathbf{Z}) - z} \, dz = 0.$$

Last step is to simplify the remaining expression by systematically use the large p, n quadratic form approximation:

$$\boxed{\vec{c}^* \mathbf{Q}(z) \vec{d} - \vec{c}^* \vec{d} \, \mathbf{g}_{\check{\mathrm{MP}}}(z) \xrightarrow[p,n\to\infty]{a.s.} 0}$$

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \widetilde{\mathbf{Q}}(z)\vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2}\mathbf{Q}(z)\boldsymbol{\Pi}^{1/2}\vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2}\mathbf{Q}\vec{\mathbf{u}}\vec{\mathbf{u}}^* \mathbf{Q}\boldsymbol{\Pi}^{1/2}\vec{\boldsymbol{a}}_N}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}\vec{\mathbf{u}}}$$

and integrate the first term

$$\frac{1}{2\boldsymbol{i}\pi}\oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2}\mathbf{Q}(z)\boldsymbol{\Pi}^{1/2}\vec{\boldsymbol{a}}_N = 0$$

Why? Because

1. the contour only encloses $\lambda_{\max}(\widetilde{\mathbf{Z}})$ which is away from the bulk,
2. but all the eigenvalues of $\mathbf{Z}$ are **in the bulk**. Hence:

$$\frac{1}{2\boldsymbol{i}\pi}\oint_{\mathcal{C}+} \frac{1}{\lambda_i(\mathbf{Z})-z}\,dz = 0.$$

Last step is to simplify the remaining expression by systematically use the large p, n quadratic form approximation:

$$\boxed{\vec{c}^*\mathbf{Q}(z)\vec{d} - \vec{c}^*\vec{d}\,\mathbf{g}_{\check{\mathrm{MP}}}(z) \xrightarrow[p,n\to\infty]{a.s.} 0} \Rightarrow \begin{cases} \vec{\boldsymbol{a}}_N^*\boldsymbol{\Pi}^{1/2}\mathbf{Q}\vec{\mathbf{u}} & \approx & \vec{\boldsymbol{a}}_N^*\boldsymbol{\Pi}^{1/2}\vec{\mathbf{u}}\,\mathbf{g}_{\check{\mathrm{MP}}}(z) \\ \vec{\mathbf{u}}^*\mathbf{Q}\boldsymbol{\Pi}^{1/2}\vec{\boldsymbol{a}}_N & \approx & \vec{\mathbf{u}}^*\boldsymbol{\Pi}^{1/2}\vec{\boldsymbol{a}}_N\,\mathbf{g}_{\check{\mathrm{MP}}}(z) \\ \vec{\mathbf{u}}^*\mathbf{Q}\vec{\mathbf{u}} & \approx & \mathbf{g}_{\check{\mathrm{MP}}}(z) \end{cases}$$

Proof VII

After simplifications,

$$\begin{aligned}\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N &\approx -\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} |\vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2} \vec{\mathbf{u}}|^2 \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)}\, dz \\ &= -\frac{\vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)}\, dz\end{aligned}$$

It remains to compute the correction factor

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)}\, dz$$

by residue calculus (not that difficult).

A **minor miracle** occurs: **This factor admits a closed form formula!**

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)}\, dz = \left(1 - \frac{c}{\theta^2}\right)\left(1 + \frac{c}{\theta}\right)^{-1}$$

Finally:

$$\boxed{\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N - \left(1 - \frac{c}{\theta^2}\right)\left(1 + \frac{c}{\theta}\right)^{-1} \vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N \xrightarrow[p,n\to\infty]{a.s.} 0\ .}$$

Summary I

Spiked model

- Let $\mathbf{\Pi}$ a small perturbation of the identity [Example: $\mathbf{\Pi} = \mathbf{I}_p + \theta \vec{\mathbf{u}}\vec{\mathbf{u}}^*$]
- $\mathbf{X}_n$ a $p \times n$ matrix with i.i.d. entries

then $\boxed{\widetilde{\mathbf{X}}_n = \mathbf{\Pi}^{1/2}\mathbf{X}_n}$ is a (multiplicative) spiked model

Global regime

Spectral measure $L_n\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$ converges to Marčenko-Pastur distribution $\mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}$

Largest eigenvalue

- if $\boxed{\theta \leq \sqrt{c}}$, then $\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$ converges to the right edge of the bulk
- if $\boxed{\theta > \sqrt{c}}$, then $\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)$ separates from the bulk

$$\lambda_{\max}\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right) \to \sigma^2(1+\theta)\left(1+\frac{c}{\theta}\right) > \sigma^2(1+\sqrt{c})^2$$

Summary II

Recall that

$$\widetilde{\mathbf{X}}_n = \mathbf{\Pi}^{1/2}\mathbf{X}_n \quad \text{where} \quad \mathbf{\Pi} = (\mathbf{I}_p + \theta \vec{\mathbf{u}}\vec{\mathbf{u}}^*) \ .$$

Eigenvector $\vec{\boldsymbol{v}}_{\max}$ associated to $\lambda_{\max}$

- Let $\vec{\boldsymbol{v}}_{\max}$ be the eigenvector associated to $\lambda_{\max}$,

$$\left(\frac{1}{n}\widetilde{\mathbf{X}}_n\widetilde{\mathbf{X}}_n^*\right)\vec{\boldsymbol{v}}_{\max} = \lambda_{\max}\vec{\boldsymbol{v}}_{\max}$$

- and $\vec{\mathbf{u}}$ the eigenvector associated to the largest eigenvalue of $\mathbf{\Pi}$

$$\mathbf{\Pi}\vec{\mathbf{u}} = (\mathbf{I}_p + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)\,\vec{\mathbf{u}} = (1+\theta)\vec{\mathbf{u}} \ .$$

Then

$$\boxed{\vec{\boldsymbol{a}}_N^*\vec{\boldsymbol{v}}_{\max}\vec{\boldsymbol{v}}_{\max}^*\vec{\boldsymbol{a}}_N - \left(1-\frac{c}{\theta^2}\right)\left(1+\frac{c}{\theta}\right)^{-1}\vec{\boldsymbol{a}}_N^*\vec{\mathbf{u}}\vec{\mathbf{u}}^*\vec{\boldsymbol{a}}_N \xrightarrow[p,n\to\infty]{a.s.} 0}$$

- Notice the **correction factor** depending on c

$$\boxed{\kappa(c) = \left(1-\frac{c}{\theta^2}\right)\left(1+\frac{c}{\theta}\right)^{-1}}$$

Lotka-Volterra system of coupled differential equations

A popular model to describe the dynamics of interacting species in foodwebs is given by a system of Lotka-Volterra equations:

$$\boxed{\frac{dx_k(t)}{dt} = x_k(r_k - x_k + (\boldsymbol{Bx})_k)} \qquad k \in [N], \quad \boldsymbol{x} = (x_k).$$

Here $(\boldsymbol{Bx})_k = \sum_\ell B_{k\ell} x_\ell$.

- N is the **number of species** in a given foodweb,
- $x_k = x_k(t)$ is the **abundance** (=population) of species k at time t,
- $\boldsymbol{r} = (r_k)$ where r_k is the **intrinsic growth rate** of species k,
- $\boldsymbol{B} = (B_{k\ell})$ where $B_{k\ell}$ is the **interaction** between species ℓ and species k

Remarks

1. if $\boldsymbol{x}_0 = \boldsymbol{x}(t=0) > 0$ then for all $t > 0$, $\boldsymbol{x}(t) > 0$.
2. if $\boldsymbol{B} = 0$ (no interactions), we recover the logistic equation

$$\frac{dx_k(t)}{dt} = x_k(r_k - x_k) \qquad \forall k \in [N]$$

Main assumption: a random model for the interaction matrix B

- The study of large Lotka-Volterra systems makes it **very difficult to calibrate** the model and **estimate** matrix B.
- An alternative is to consider random matrices, **the statistical properties** of which **encode some real properties of the foodwed**.
- it is a very rough approach but we need a model otherwise ..

$$\boxed{\text{No maths} = \text{no understanding}}$$

A. Rossberg, in Food webs and biodiversity (Wiley)

Some random models

- **The Wigner model, (real) Ginibre model**: poor adequation to reality but a good benchmark to explore the mathematical tractability
- **The elliptic model:** encodes the natural correlation between $B_{k\ell}$ and $B_{\ell k}$
- **Sparse models:** encodes the fact that a species only interacts with $d \ll N$ other species.
- **Variance profiles, non-centered matrices**, etc: of interest but harder to analyze.

Questions

Recall the model of interest

$$\boxed{\frac{dx_k(t)}{dt} = x_k(r_k - x_k + (\boldsymbol{B}\boldsymbol{x})_k)} \qquad \boldsymbol{x} = (x_k)\,.$$

- **Existence of an equilibrium $\boldsymbol{x}^\star = (x_k^\star)$** such that

$$x_k^\star(r_k - x_k^\star + (\boldsymbol{B}\boldsymbol{x}^\star)_k) = 0 \quad \forall k \in [N]\,.$$

- **Stability** of this equilibrium: if $\boldsymbol{x}_0 > 0$, do we have

$$\boldsymbol{x}(t) \xrightarrow[t\to\infty]{} \boldsymbol{x}^\star ?$$

- **Properties** of the equilibrium $\boldsymbol{x}^\star$, in particular **species extinction**:

$$\text{Can we have } x_k^\star = 0 \quad \text{for some} \quad k \in [N] \ ?$$

- **Feasibility** of this equilibrium:

$$\text{Can we have } x_k^\star > 0 \quad \text{for all} \quad k \in [N]?$$

- Existence of **Multiple equilibria**?

The typical dynamics of a LV system

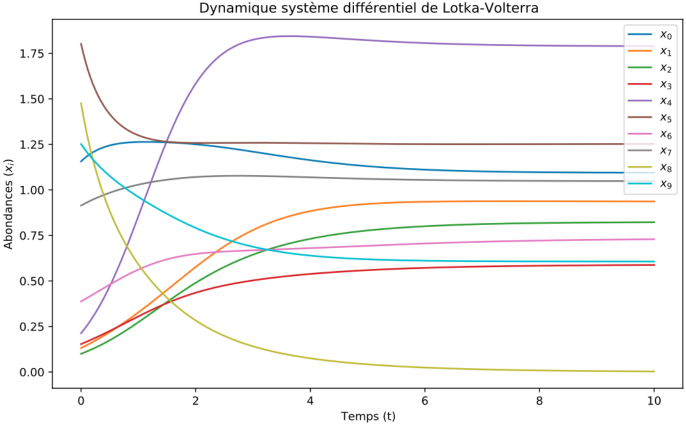


Figure: A LV system with 9 species, converging to equilibrium, with one species vanishing

The elliptic model for Large LV systems

- We will mainly consider the **ρ-elliptic** model where

$$\begin{pmatrix} X_{ij} \\ X_{ji} \end{pmatrix} \sim \mathcal{N}_2 \left(\mathbf{0}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

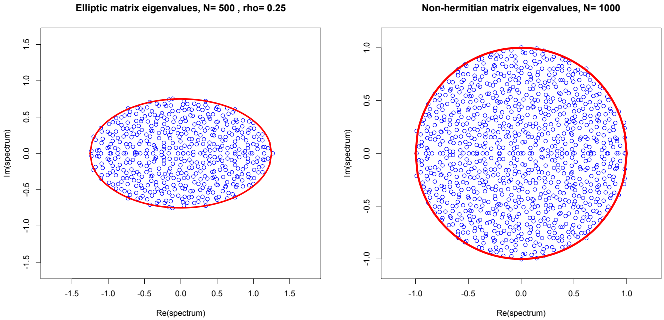


Figure: elliptic law (left) – circular law corresponding to $\rho = 0$ (right)

Remark

- Notice that this model interpolates between the (real) Ginibre model where $\rho = 0$ and the Wigner model where $\rho = 1$.

Stability

Recall the model of interest

$$\boxed{\frac{dx_k(t)}{dt} = x_k(r_k - x_k + (\boldsymbol{B}\boldsymbol{x})_k)} \quad \text{where} \quad \boldsymbol{B} = \frac{X}{\boldsymbol{\kappa}\sqrt{N}}$$

and

- X is ρ-elliptic and the spectrum of $\frac{X}{\sqrt{N}}$ is of order $\mathcal{O}(1)$,
- $\boldsymbol{\kappa}$ is an extra normalizing parameter.

A sufficient condition for stability

Suppose that $\boldsymbol{B} = \frac{X}{\boldsymbol{\kappa}\sqrt{N}}$ where X is $N \times N$ random ρ-elliptic. If

$$\boxed{\boldsymbol{\kappa} > \sqrt{2(1+\rho)}}$$

then the LV system **a.s. eventually** admits a unique stable equilibrium $\boldsymbol{x}_N^\star \geq 0$.

Elements of proof

- Based on a sufficient result of Takeuchi and Adachi (1980)
- the existence of a Lyapounov function
- the asymptotic behaviour of

$$\lambda_{\max}(B + B^T) = \frac{1}{\boldsymbol{\kappa}} \lambda_{\max}\left(\frac{X + X^T}{\sqrt{N}}\right)$$

Feasible equilibrium: a simple linear equation

Consider the LV system

$$\boxed{\frac{dx_k}{dt} = x_k(r_k - x_k + (\boldsymbol{B}\boldsymbol{x})_k)} \qquad \text{where} \qquad \boldsymbol{B} = \frac{\boldsymbol{X}}{\boldsymbol{\kappa}\sqrt{N}}$$

- We investigate the case where there exists a **positive equilibrium**

$$\boldsymbol{x}^* > 0 \quad \Leftrightarrow \quad x_k^* > 0 \quad \forall k \in [N]\,.$$

- In theoretical ecology it is called a **feasible equilibrium** and is of interest because all species survive.
- Such an equilibrium should satisfy

$$r_k - x_k^* + (\boldsymbol{B}\boldsymbol{x}^*)_k = 0 \quad \Leftrightarrow \quad \boxed{\boldsymbol{x}^* = \boldsymbol{r} + \boldsymbol{B}\boldsymbol{x}^*}\,, \quad \boldsymbol{x}^* > 0.$$

- If matrix $I - \boldsymbol{B}$ is invertible, then

$$\boldsymbol{x}^* = (I - \boldsymbol{B})^{-1}\boldsymbol{r}\,.$$

A logarithmic correction implies feasibility

Suppose that X is $\boldsymbol{\rho}$-elliptic and consider the system

$$\boldsymbol{x}^* = \mathbf{1} + \frac{\boldsymbol{X}}{\boldsymbol{\kappa}\sqrt{N}}\boldsymbol{x}^* \quad \text{where} \quad \boldsymbol{\kappa} = \boldsymbol{\kappa}_N \xrightarrow[N\to\infty]{} \infty\,.$$

Denote by $\boxed{\boldsymbol{\kappa}_N^* = \sqrt{2\log(N)}}$.

Theorem (phase transition)

- If $\boldsymbol{\kappa}_N \leq (1-\delta)\boldsymbol{\kappa}_N^*$ for $N \gg 1$ then $\boxed{\mathbb{P}\left\{\inf_{k\in[N]} x_k^* > 0\right\} \xrightarrow[N\to\infty]{} 0}$
- If $\boldsymbol{\kappa}_N \geq (1+\delta)\boldsymbol{\kappa}_N^*$ for $N \gg 1$ then $\boxed{\mathbb{P}\left\{\inf_{k\in[N]} x_k^* > 0\right\} \xrightarrow[N\to\infty]{} 1}$

References

- Bizeul-N., **2021** (Ginibre case), Clenet, El Ferchichi, N. **2022** (Elliptic case).

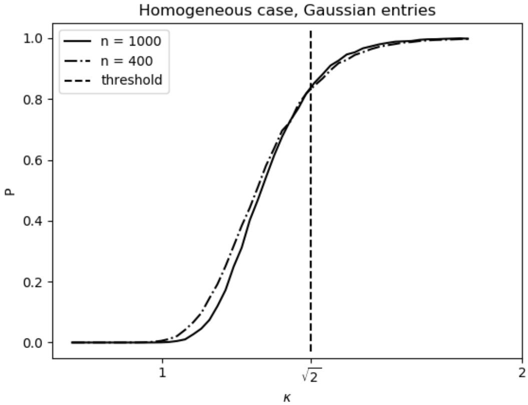
About the logarithmic factor

- Notice that
$$\left\| \frac{\boldsymbol{X}}{\kappa_N^* \sqrt{N}} \right\| = \mathcal{O}\left(\frac{1}{\sqrt{2\log(N)}} \right)$$
Hence
$$\begin{aligned} \boldsymbol{x}^* &= \mathbf{1} + \frac{\boldsymbol{X}}{\boldsymbol{\kappa}\sqrt{N}} \boldsymbol{x}^* \\ &\simeq \mathbf{1} \end{aligned}$$
as $N \to \infty$.
- but

N	10^2	10^3	10^4	10^5	10^6
$\frac{1}{\boldsymbol{\kappa}_N^*}$	0.33	0.27	0.23	0.21	0.19

- the logarithmic factor **decreases extremely slowly** to zero.

Phase transition



- We plot the frequency of positive solutions over 10.000 trials for the system

$$\boldsymbol{x}^* = \mathbf{1} + \frac{1}{K\sqrt{\log(N)}} \frac{\boldsymbol{X}}{\sqrt{N}} \boldsymbol{x}^*$$

  as a function of the parameter K.
- A phase transition occurs at the critical value $K = \sqrt{2}$.

Statistical properties of the equilibrium

Existence of an equilibrium

Recall that if $\kappa > \sqrt{2(1+\rho)}$, then

- a.s. eventually there exists a stable equilibrium $\boldsymbol{x}_N^\star$ to the LV system

$$\frac{dx_k(t)}{dt} = x_k(r_k - x_k + (\boldsymbol{B}\boldsymbol{x})_k)\,, \quad k \in [N]$$

Question

Matrix $\boldsymbol{B}$ being random, so is $\boldsymbol{x}^\star$,

- How can we extract statistical information on $\boldsymbol{x}^\star$ from this?
- We focus on the **empirical measure of the equilibrium**

$$\boxed{\mu^\star = \frac{1}{N}\sum_{i\in[N]} \delta_{x_i^\star}}$$

in the asymptotic regime $N \to \infty$.

Main results I

- Recall the LV system $\boxed{\dfrac{dx_k(t)}{dt} = x_k(r_k - x_k + (\boldsymbol{B}\boldsymbol{x})_k)}$ where $\boldsymbol{B} = \dfrac{\boldsymbol{X}}{\boldsymbol{\kappa}\sqrt{N}}$
- Assume that $\boldsymbol{r} \perp\!\!\!\perp \boldsymbol{B}$ and $\quad (a.s.) \quad \mu^{\boldsymbol{r}} \xrightarrow[N\to\infty]{w, L^2} \mathcal{L}(\bar{r})\,.$

Theorem (Gueddari-Hachem-N. , 2024)

- Let $\boldsymbol{\kappa} > \sqrt{2(1+\rho)}$, then

$$\boxed{\mu^\star \xrightarrow[N\to\infty]{w, L^2} \quad \mathcal{L}\left((1+\rho\gamma/\delta^2)\left(\sigma\bar{Z}+\bar{r}\right)_+\right)} \qquad \bar{Z}\sim\mathcal{N}(0,1)\,, \quad \bar{Z}\perp\!\!\!\perp \bar{r}\,.$$

where (δ, σ, γ) satisfy

$$\begin{aligned} \boldsymbol{\kappa} &= \delta + \rho\frac{\gamma}{\delta}\,, \\ \sigma^2 &= \frac{1}{\delta^2}\mathbb{E}\left(\sigma\bar{Z}+\bar{r}\right)_+^2\,, \qquad \bar{Z}\sim\mathcal{N}(0,1)\,, \quad \text{indep. from } \bar{r} \\ \gamma &= \mathbb{P}\Big[\sigma\bar{Z}+\bar{r}>0\Big]. \end{aligned}$$

Consequences

Proportion of surviving species

We are interested in the proportion of **surviving species**, depending on κ:

$$\boxed{\frac{1}{N}\sum_{i\in[N]} \mathbf{1}_{\{x_i^\star>0\}} \quad \simeq \quad \mathbb{P}\left(\sigma\bar{Z}+\bar{r}>0\right)=\gamma}$$

Distribution of surviving species

Denote by $\boldsymbol{s}(\boldsymbol{x}^\star)$ the subvector of $\boldsymbol{x}^\star$ with positive components of $\boldsymbol{x}^\star$. Its size $|\boldsymbol{s}(\boldsymbol{x}^\star)|$ is random and the empirical distribution of the surviving species is:

$$\mu^{\boldsymbol{s}(\boldsymbol{x}^\star)} = \frac{1}{|\boldsymbol{s}(\boldsymbol{x}^\star)|}\sum_{i\in[|\boldsymbol{s}(\boldsymbol{x}^\star)|]} \delta_{[\boldsymbol{s}(\boldsymbol{x}^\star)]_i}\ .$$

A good proxy should be

$$\boxed{\mathcal{L}\left((1+\rho\gamma/\delta^2)\left(\sigma\bar{Z}+\bar{r}\right)_+ \,\middle|\, \sigma\bar{Z}+\bar{r}>0\right)\ ,}$$

the density of which is explicit (often referred to as "truncated Gaussian").

References

- Bunin **2017**, Galla **2018** (based on theoretical physics methods)
- Akjouj, Hachem, Maïda, N. **2023**, ArXiv. (Wigner case)
- Gueddari, Hachem, N. **2024**, Arxiv. (Elliptic case)

Simulations I: Proportion of surviving species

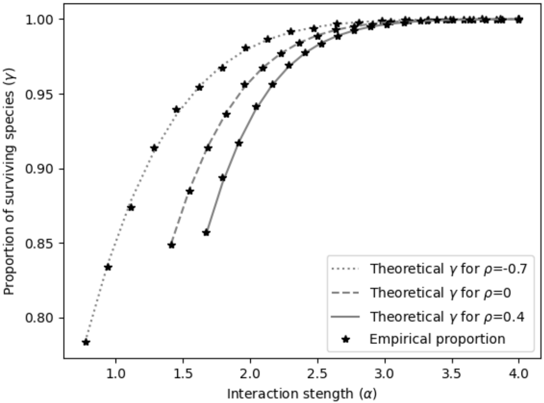


Figure: Experimental proportions of surviving species vs theoretical values γ for three correlation coefficients $\rho = -0.7, 0, 0.4$ w.r.t. the interaction strength (κ).

Simulations II: density of surviving species

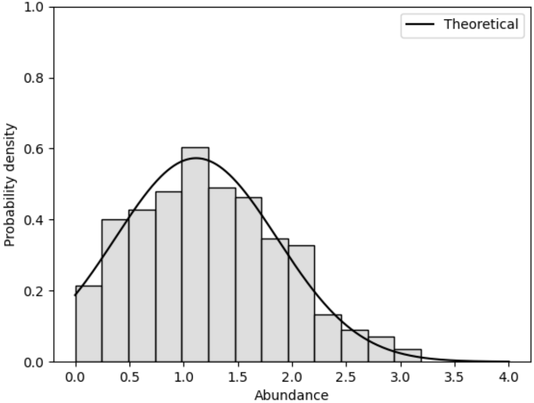


Figure: Histogram of positive abundances vs the truncated gaussian density function f_{surv} for $\rho = 0.4$ with the interaction strength fixed to $\kappa = 2$.

Elements of proof

We can prove that an equilibrium is defined by the following set of constraints:

$$\begin{cases} \boldsymbol{x}^\star & \geq \quad 0\,, \\ (I-\boldsymbol{B})\boldsymbol{x}^\star - \boldsymbol{r} & \geq \quad 0\,, \\ x_i^\star\left([(I-\boldsymbol{B})\boldsymbol{x}^\star]_i - r_i\right) & = \quad 0\,. \end{cases}$$

We can get an alternative, fixed-point equation formulation: Let $x_+ = \max(x,0)$. Then

$$\boxed{\boldsymbol{z} = \boldsymbol{B}\boldsymbol{z}_+ + \boldsymbol{r}}$$

if and only if $\boldsymbol{z}_+ = \boldsymbol{x}^\star$.

Remarks

- There are many procedures to compute a fixed-point. For example

$$\begin{cases} \boldsymbol{z}_0 \in \mathbb{R}^N \\ \boldsymbol{z}_{p+1} = \boldsymbol{B}(\boldsymbol{z}_p)_+ + \boldsymbol{r} \end{cases}.$$

- The main issue is to keep track of the distribution of $\boldsymbol{z}_p$.
- Notice the strong dependence between the $\boldsymbol{z}_p$'s in the previous scheme.
- The class of **approximate message passing algorithms** (developed by Montanari et al.) is well suited to fulfill this task.

Beyond a unique equilibrium

- So far, we have presented some progress to understand large LV systems at a mathematical level of understanding.
- The truth is that we keep on running after theoretical physicists

  Altieri, Barbier, Biroli, Bunin, Cammarota, Galla, Ros, etc.

  who discovered these formulas and many other phenomenas earlier, relying on powerful cavity, replicas, free energy computation methods, etc. many of which inspired from Parisi's work.

Beyond a unique equilibrium

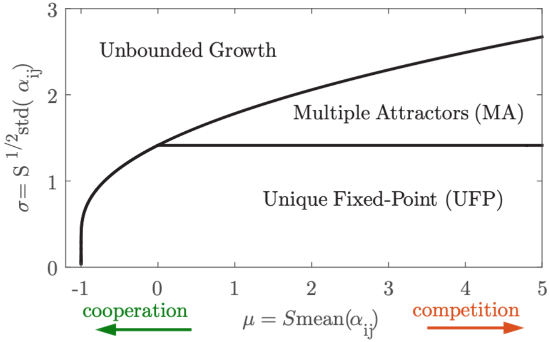


Figure: A figure taken from Bunin's 2017 article "Ecological communities with Lotka-Volterra dynamics" associated to the model

- In particular Bunin and other physicists predict phases with multiple equilibria for large LV models.
- It is an open question to understand this mathematically.

Overall conclusion

Thank you for your attention!

Introduction

Basic technical means

Wigner's theorem

Large Covariance Matrices

Spiked models

Large Lotka-Volterra systems of ODE

Appendix