Analyzing dependent data with vine copulas (Lecture 1)

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École d'été 2018 Sept 2-5,2018

Outline

What to expect:

There are about 1600 items found in google scholar with the search expression vine copula, over 200 of them are in 2018. A word cloud with the 20 most used words in articles in 2018 shows

Abalone data set

The abalone dataset is available from the University of California Irvine (UCI) machine learning repository
http://archive.ics.uci.edu/ml/datasets/Abalone

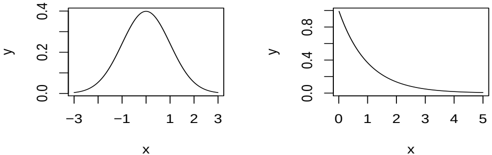
- Sex / nominal / – / M, F, and I (infant)
- Length / continuous / mm / Longest shell measurement
- Diameter / continuous / mm / perpendicular to length
- Height / continuous / mm / with meat in shell
- Whole weight / continuous / grams / whole abalone
- Shucked weight / continuous / grams / weight of meat
- Viscera weight / continuous / grams / gut weight (after bleeding)
- Shell weight / continuous / grams / after being dried
- Rings / integer / – / $+1.5$ gives the age in years

Overview

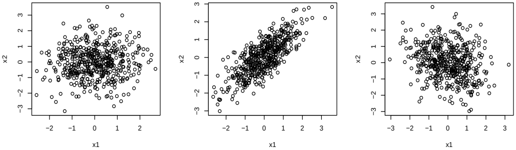
Outline

Multivariate distributions

- Multivariate distributions describe stochastic behavior of several variables jointly.
- Marginal distributions describe stochastic behavior of a single variable (examples: univariate normal, exponential)



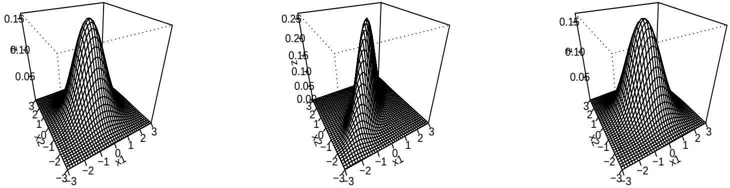
multivariate normal ($\rho = 0,\ \rho = .8, \rho = -.5$)



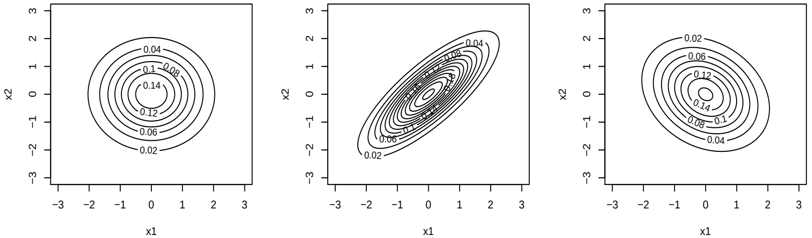
How to construct multivariate distributions with different margins?

Bivariate normal density and contour plots

joint density plot (right: $\rho = 0$, middle: $\rho = .8$, left: $\rho = -.25$)



contour plot



Conditional distributions

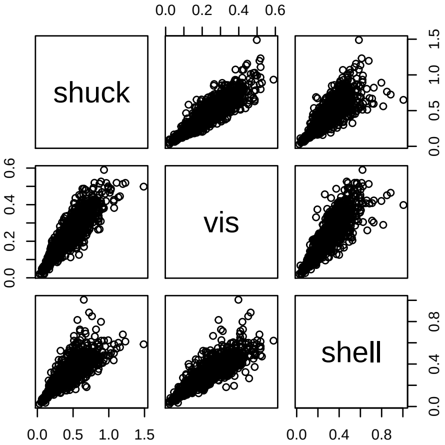
- vine distributions are defined using conditional distributions
- conditional distributions describe the stochastic behaviour of variables under the condition that other variables are fixed.
- conditional = unconditional distributions if variables are independent
- **conditional cdf of X_i given $X_j = x_j$:**

$$F_{i|j}(x_i|x_j) = \frac{1}{f_j(x_j)} \frac{\partial}{\partial x_j} F_{ij}(x_i, x_j)$$

- **conditional pdf of (X_i, X_j) given that $X_k = x_k$:**

$$f_{i,j|k}(x_i, x_j|x_k) := \frac{f_{ijk}(x_i, x_j, x_k)}{f_k(x_k)}$$

Weight variables in Abalone data



Dependency measures

- Most well known dependency measure is the correlation ρ between two random variables.
- It only measures linear dependencies.
- Non linear dependencies can be detected by Kendall's τ which measures the difference between the concordance and discordance probability.
- Upper (lower) tail dependence measures the probability of joint large (small) occurrences as one moves to the extremes.
- multivariate normal has no tail dependence, while the multivariate t distribution has tail dependence.
- When upper and lower tail dependence are not the same we speak of asymmetric tail dependence.

How to separate dependency patterns from the marginal behavior?

Outline

Resources for copulas

- Copula theory started with Sklar (1959)
- **Books on copulas:**
  - Multivariate models and multivariate dependence concepts by (Joe 1997)
  - An Introduction to Copulas (Nelsen 2006)
  - Simulating copulas: stochastic models, sampling algorithms, and applications (2nd edition) (Scherer and Mai 2017)
- **Software:**
  - copula (Kojadinovic and Yan 2010)
  - VineCopula (Schepsmeier et al. 2017)

Copula approach

Consider d random variables $\mathbf{X} = (X_1, \ldots, X_d)$ with

	pdf	**cdf**
marginal	$f_i(x_i), i = 1, \ldots, d$	$F_i(x_i), i = 1, \ldots, nd$
joint	$f(x_1, \ldots, x_d)$	$F(x_1, \ldots, x_d)$
conditional	$f(\cdot\|\cdot)$	$F(\cdot\|\cdot)$

Copula (distribution)

A copula $C(u_1, \ldots, u_d)$ is a multivariate distribution on $[0, 1]^d$ with uniformly distributed marginals.

Sklar's theorem (1959)

A joint distribution function F with margins $F_j, j = 1, \ldots, d$ can be expressed as

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \tag{1}$$

for some copula C. It is unique when F is absolutely continous.

Sklar's theorem for (conditional) densities

Densities and conditional densities for d=2

Let f_{12} denote the density of the bivariate distribution F_{12}, then

$$\begin{aligned} f_{12}(x_1, x_2) &= c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) \cdot f_2(x_2) \qquad (2) \\ f_{2|1}(x_2|x_1) &= c_{12}(F_1(x_1), F_2(x_2)) \cdot f_2(x_2) \end{aligned}$$

where $c_{12}(\cdot)$ is the density assiociated with the copula C_{12}.

Equations (1) and (2) can also be used in a constructive way to build new multivariate distributions.

Variable scales

Let $(X_1, \ldots, X_d) \sim F$.

- **x-scale**: original scale of variables $X_j, j = 1, \ldots, d$
- **u-scale**: copula scale of variables $U_j := F_j(X_j) \sim U(0,1), j = 1, \ldots, d$
- **z-scale**: normalized score scale of variables $Z_j := \Phi^{-1}(U_j) \sim N(0,1), j = 1, \ldots, d$

For $d = 2$ pair plots and density contour plots on

- x-scale mix dependence and marginal effects
- u-scale show dependence effects, but not so informative
- z scale show dependence effects and can be compared to known behavior of bivariate normal variables. The associated contour plot is called a normalized contour plot

Bivariate copula families

- **Elliptical copulas:** Copula constructed using inversion of Sklar's theorem applied to bivariate elliptical distributions such as the bivariate normal or Student t distribution.
- **Archemedian copulas:** Copulas directly constructed using a strictly monotone convex generator function ψ with $\psi(0) = 0$

$$C(u_1, u_2) = \psi^{[-1]}(\psi(u_1) + \psi(u_2)), \tag{3}$$

  where $\psi^{[-1]}$ denotes the pseudo inverse of ψ. Examples are Gumbel, Clayton and Frank copulas.
- **Extreme value copulas**: This class of copulas associated with limiting distributions of bivariate extreme value theory. An interesting nonsymmetric class is the Tawn copula with 3 parameters.

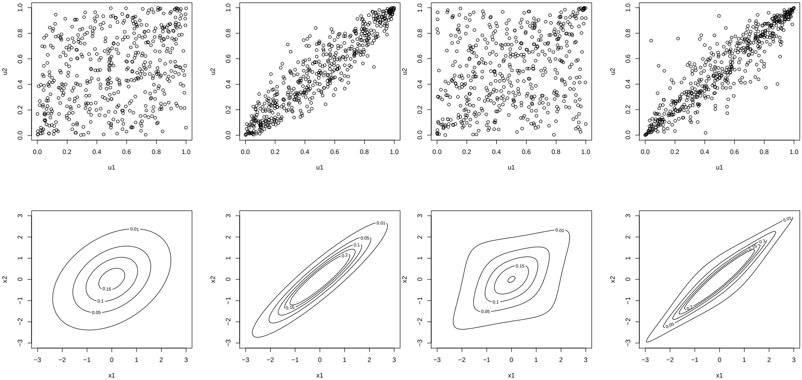
Bivariate elliptical copula families

Gaussian copula

(left $\tau = .25$, right: $\tau = .75$)

t-copula with $df = 3$

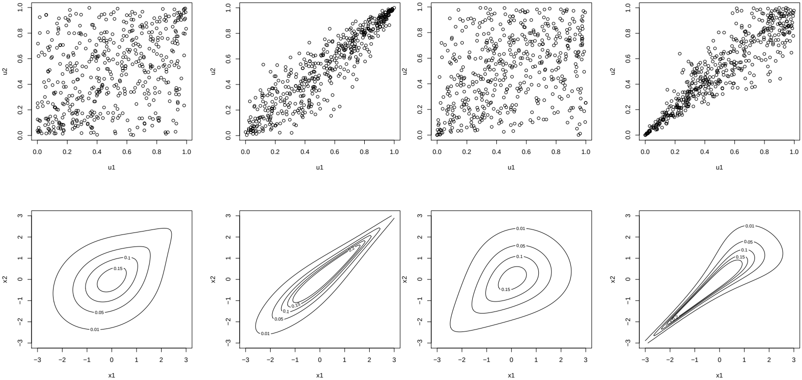
(left $\tau = .25$, right: $\tau = .75$)



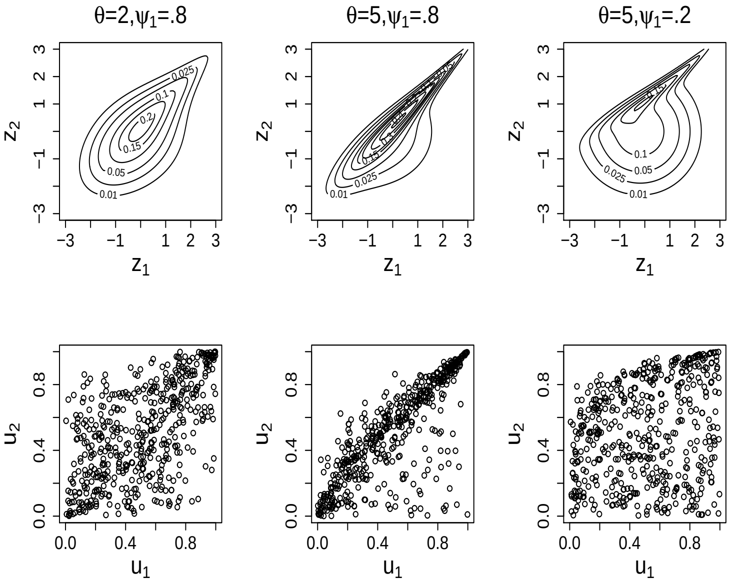
Bivariate Archimedian copula families

Gumbel copula
(left $\tau = .25$, right: $\tau = .75$)

Clayton copula
(left $\tau = .25$, right: $\tau = .75$)



Bivariate Tawn copula with 2 parameters



Some formulas of bivariate copulas (part 1)

- **Gaussian copula:**

$$C(u_1, u_2; \rho) = \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho),$$

where $\Phi(\cdot)$ is $N(0,1)$ cdf and $\Phi_2(\cdot, \cdot; \rho)$ is bivariate normal cdf with zero means, unit variances and correlation ρ. The pdf is

$$c(u_1, u_2; \rho) = \frac{1}{\phi(x_1)\phi(x_2)} \frac{1}{\sqrt{1-\rho^2}} \exp\left\{ -\frac{\rho^2(x_1^2 + x_2^2) - 2\rho x_1 x_2}{2(1-\rho^2)} \right\},$$

where $x_1 := \Phi^{-1}(u_1)$ and $x_2 := \Phi^{-1}(u_2)$.

- **Student t copula:**

$$c(u_1, u_2; \nu, \rho) = \frac{t(T_\nu^{-1}(v_1), T_\nu^{-1}(v_2); \nu, \rho)}{t_\nu(T_\nu^{-1}(v_1))t_\nu(T_\nu^{-1}(v_2))},$$

where $T_\nu(t_v)$ are univariate Student t cdf (pdf) with $df = \nu$ and $t(\cdot, \cdot; \nu, \rho)$ pdf of bivariate Student t with $df = \nu$ and scale matrix $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.

Some formulas of bivariate copulas (part 2)

- **Clayton copula:**

$$C(u_1, u_2) = (u_1^{-\delta} + u_2^{-\delta} - 1)^{-\frac{1}{\delta}},$$

where $0 < \delta < \infty$, $\delta \to 0$ corresponds to independence

- **Gumbel copula:**

$$C(u_1, u_2) = \exp[-\{(-\ln u_1)^{\delta} + (-\ln u_2)^{\delta}\}^{\frac{1}{\delta}}],$$

where $\delta \geq 1$, $\delta = 1$ corresponds to independence.

Kendall's τ of some copula families

Kendall's tau

The Kendall's τ between X_1 and X_2 is defined as

$$\tau := P((X_{11} - X_{21})(X_{12} - X_{22}) > 0) - P((X_{11} - X_{21})(X_{12} - X_{22}) < 0),$$

where (X_{11}, X_{12}) and (X_{21}, X_{22}) are i.i.d copies of (X_1, X_2).
Further for the associated copula C we can express

$$\tau = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2).$$

Family	**Kendall's τ**
Gaussian	$\tau = \frac{2}{\pi} arcsin(\rho)$
Student t	$\tau = \frac{2}{\pi} arcsin(\rho)$
Clayton	$\tau = \frac{\delta}{\delta+2}$
Gumbel	$\tau = 1 - \frac{1}{\delta}$

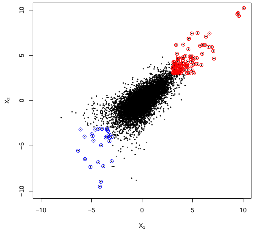
Bivariate tail dependence

Upper and lower tail dependence coefficient

$$\lambda^{upper} = \lim_{t \to 1^-} P(X_2 > F_2^{-1}(t) | X_1 > F_1^{-1}(t)) = \lim_{t \to 1^-} \frac{1 - 2t + C(t,t)}{1 - t},$$

$$\lambda^{lower} = \lim_{t \to 0^+} P(X_2 \le F_2^{-1}(t) | X_1 \le F_1^{-1}(t)) = \lim_{t \to 0^+} \frac{C(t,t)}{t}.$$

Illustration: upper tail (red), lower tail (blue)



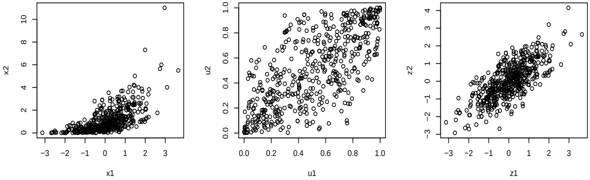
Tail dependence of bivariate copula families

Family	Upper tail	Lower tail
Gaussian	$\lambda^{up} = 0$	$\lambda^{low} = 0$
Student t	$\lambda^{up} = 2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$	$\lambda^{low} = 2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$
Clayton	$\lambda^{up} = 0$	$\lambda^{low} = 2^{-1/\delta}$
Gumbel	$\lambda^{up} = 2 - 2^{1/\delta}$	$\lambda^{low} = 0$

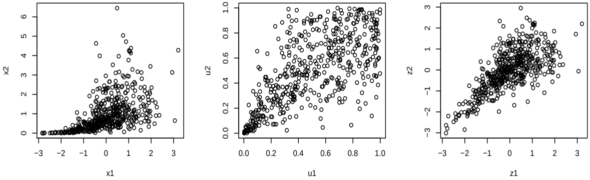
Meta distributions

are build using a copula $[(u_1, u_2)]$ and different margins (normal/exponential $[(x_1, x_2)]$ or normal/normal $[(z_1, z_2)]$)

Gaussian copula



Clayton copula



Bivariate rotations

- **Extension:** To extend range of dependence we use counterclockwise rotations
  - **90 degree:** $c_{90}(u_1, u_2) := c(1 - u_2, u_2)$
  - **180 degree:** $c_{180}(u_1, u_2) := c(1 - u_1, 1 - u_2)$
  - **270 degree:** $c_{270}(u_1, u_2) := c(u_2, 1 - u_1)$
- **Extended Clayton:**

$$c_{clayton}^{extended}(u_1, u_2; \delta) := \begin{cases} c_{clayton}(u_1, u_2) & \text{if } \delta > 0 \\ c_{clayton}(1 - u_2, u_1) & \text{otherwise} \end{cases}$$

- **Exchangeability or reflection symmetry:**

$$c(u_1, u_2) = c(u_2, u_1) \text{ for all } u_1, u_2$$

  - Gumbel and Clayton are exchangeable
  - 90 or 270 degree rotation is no longer exchangeable

Illustration of rotations

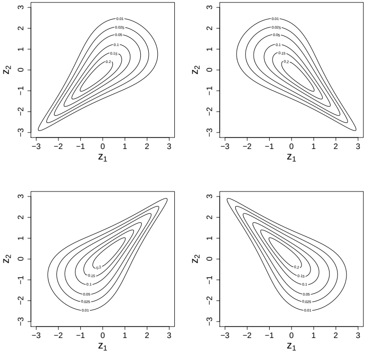


Figure: Normalized contour plots of Clayton rotations: top left: 0 degree rotation ($\tau = .5$), top right: 90 degree rotation ($\tau = -.5$), bottom left: 180 degree rotation ($\tau = .5$), bottom right: 270 degree rotation ($\tau = -.5$).

Parametric bivariate copula models

- **Data:** i.i.d observations $(x_{i1}, x_{i2}), i = 1, \cdots n$ from the joint density $f_{12}(x_1, x_2) = c_{12}(F_1(x_1), F_2(x_2))f_1(x_1)f_2(x_2)$.
- **Margins**: $F_j(x_j; \boldsymbol{\theta}_{mj}), j = 1, 2$ with marginal parameters $\boldsymbol{\theta}_m = (\boldsymbol{\theta}_{m1}, \boldsymbol{\theta}_{m2})$.
- **Copula**: $c_{12}(u_1, u_2; \boldsymbol{\theta}_c)$ with copula parameter $\boldsymbol{\theta}_c$.
- **Estimation:**
  - **Joint:** Marginal and copula parameter are jointly estimated using maximum likelihood (ML).
  - **Two step:**
    - **Inference for margins:** Estimate margin separately to get $\hat{F}_j^{par}$ and then use ML based on $\hat{u}_{ij}^{par} = \hat{F}_j^{par}(x_{ij})$. Joe and Xu (1996)
    - **Semiparametric approach:** Estimate margins using empirical cdf's $\hat{F}_j$ and then use ML based on $\hat{u}_{ij} = \hat{F}_j(x_{ij})$.Genest et al. (1995)

Nonparametric bivariate copula models

- **Data:** i.i.d observations $(x_{i1}, x_{i2}), i = 1, \cdots n$ from the joint density $f_{12}(x_1, x_2) = c_{12}(F_1(x_1), F_2(x_2))f_1(x_1)f_2(x_2)$.
- **Models:** Both marginal and copula models are not specified
- **Two step estimation:**
  - Margins are estimated using empirical cdf's $\hat{F}_j$. The empirical copula is estimated based $\hat{u}_{ij} = \hat{F}_j(x_{ij})$.
  - Margins are estimated using kernel density cdf estimates $\hat{F_j^{kd}}$ and copula density is estimated by bivariate kernel estimates based on $\hat{u}_{ij}^{ks} = \hat{F}_j^{ks}(x_{ij})$.

General model selection criteria

Let $\ell_n(\hat{\boldsymbol{\theta}}, \mathbf{x})$ be the log likelihood based on model with p dimensional parameter $\boldsymbol{\theta}$ and observed data $\mathbf{x}$ of size n evaluated at the estimate $\hat{\boldsymbol{\theta}}$.

AIC: (Akaike 1973)

$$AIC_n := 2\ell_n(\hat{\boldsymbol{\theta}}, \mathbf{x}) + 2p$$

BIC: (Schwarz 1978)

$$BIC_n := 2\ell_n(\hat{\boldsymbol{\theta}}, \mathbf{x}) + \log(n)p$$

Bivariate copula estimation in VineCopula

The R package VineCopula allows also for bivariate copula estimation

Function	Purpose
BiCop	Constructing BiCop-objects
BiCopCDF	Distribution Function of a Bivariate Copula
BiCopCondSim	Conditional simulation from a Bivariate Copula
BiCopEst	Parameter Estimation for Bivariate Copula Data
BiCopEstList	List of Maximum Likelihood Estimates for Several Bivariate Copula Families
BiCopGofTest	Goodness-of-Fit Test for Bivariate Copulas
BiCopHfunc	Conditional Distribution Function of a Bivariate Copula
BiCopKDE	Kernel estimate of a Bivariate Copula Density
BiCopMetaContour	Contour Plot of Bivariate Meta Distribution
BiCopPar2TailDep Tail	Dependence Coefficients of a Bivariate Copula
BiCopPar2Tau	Kendall's Tau Value of a Bivariate Copula
BiCopPDF	Density of a Bivariate Copula
BiCopSelect	Selection and Maximum Likelihood Estimation of Bivariate Copula Families
BiCopSim	Simulation from a Bivariate Copula
BiCopTau2Par	Parameter of a Bivariate Copula for a given Kendall's Tau Value

Outline

Motivation for vine based models

- Many data structures exhibit
  - different marginal distributions
  - non-symmetric dependencies between some pairs of variables
  - heavy tail dependencies between some pairs of variables
- These cannot be modeled with a Gaussian or multivariate t distribution
- The copula approach allows to model dependencies and marginal distributions separately.
- Marginal time dependencies can be captured by appropriate univariate time series models.
- Elliptical and Archimedean copulas do not allow for different dependency patterns between pairs of variables.

Vine based models can overcome all these shortcomings.

Some (notational) remarks

- We distinguish between the copula associated with a bivariate conditional distribution and a bivariate conditional distribution derived from the copula variables. In particular
  - The conditional distribution of (X_i, X_j) given $\mathbf{X_D} = \mathbf{x_d}$ has copula $C_{ij;D}(\cdot,\cdot)$. We call $C_{ij;D}(\cdot,\cdot)$ a conditional copula.
  - Assuming that $(U_1, \ldots, U_d)$ have the copula C as distribution function, the bivariate distribution of (U_i, U_j) given $\mathbf{U_d} = \mathbf{u_d}$ is denoted by $C_{ij|D}(\cdot,\cdot)$. This is in general not a copula.
- Specification of three bivariate copulas does not lead in general to a valid construction of three variate copula.

Pair-copula constructions in 3 dimensions

$$f(x_1, x_2, x_3) = f_{3|12}(x_3|x_1, x_2) f_{2|1}(x_2|x_1) f_1(x_1)$$

Using Sklar for $f(x_1, x_2)$, $f(x_2, x_3)$ and $f_{13|2}(x_1, x_3|x_2)$ implies

$$\begin{aligned}
f_{2|1}(x_2|x_1) &= c_{12}(F_1(x_1), F_2(x_2)) f_2(x_2) \\
f_{3|12}(x_3|x_1, x_2) &= c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) f_{3|2}(x_3|x_2) \\
&= c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) c_{23}(F_2(x_2), F_3(x_3)) f_3(x_3)
\end{aligned}$$

$$\begin{aligned}
f(x_1, x_2, x_3) &= c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) c_{23}(F_2(x_2), F_3(x_3)) \\
&\times c_{12}(F_1(x_1), F_2(x_2)) \\
&\times f_3(x_3) f_2(x_2) f_1(x_1)
\end{aligned}$$

Only bivariate copulas and univariate conditional cdf's are used.
We will later generalize this to d dimensions.

Parametric copula; simplifying assumption

- The bivariate copulas in occuring in a PCC are called pair copulas.
- Pair copulas can be parametrically modelled with parameter $\boldsymbol{\theta}$, i.e. we write $C_{ij}(\cdot,\cdot;\boldsymbol{\theta})$.
- The conditional copulas $C_{ij;D}$ dependgenerally on the conditioning value $\mathbf{x_D}$, we therefore use $C_{ij;D}(\cdot,\cdot;\mathbf{x}_D)$.

- **Simplifying assumption**

  If there is no dependency, i.e.

  $$C_{ij;D}(\cdot,\cdot;\mathbf{x}_D) = C_{ij;D}(\cdot,\cdot) \text{ for all } \mathbf{x_D},$$

  we say that the simplifying assumption holds.

Simplified PCC's in 3 dimensions

In the PCC we can reorder the variables, therefore we get three PCC's .

Three simplified PCC's in 3 dimensions

$C_{12} - C_{23} - C_{13;2}$:

$$\begin{aligned} f(x_1,x_2,x_3) &= c_{13;2}(F_{1|2}(x_1|x_2),F_{3|2}(x_3|x_2))c_{12}(F_1(x_1),F_2(x_2)) \\ &\times c_{23}(F_2(x_2),F_3(x_3))f_1(x_1)f_2(x_2)f_3(x_3) \end{aligned}$$

$C_{13} - C_{23} - C_{12;3}$:

$$\begin{aligned} f(x_1,x_2,x_3) &= c_{12;3}(F_{1|3}(x_1|x_3),F_{2|3}(x_2|x_3))c_{13}(F_1(x_1),F_3(x_3)) \\ &\times c_{23}(F_2(x_2),F_3(x_3))f_1(x_1)f_2(x_2)f_3(x_3) \end{aligned}$$

$C_{12} - C_{13} - C_{23;1}$:

$$\begin{aligned} f(x_1,x_2,x_3) &= c_{23;1}(F_{2|1}(x_2|x_1),F_{3|1}(x_3|x_1))c_{12}(F_1(x_1),F_2(x_2)) \\ &\times c_{13}(F_1(x_1),F_3(x_3))f_1(x_1)f_2(x_2)f_3(x_3) \end{aligned}$$

Storing the PCC with matrices

- $c_{12} - c_{23} - c_{13;2}$:

$$\text{Mat:=}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix} \text{ and Fam:=}\begin{bmatrix} 0 & 0 & 0 \\ f_{13;2} & 0 & 0 \\ f_{13} & f_{23} & 0 \end{bmatrix}.$$

  - Column 1 of Mat identifies copulas $c_{13;2}$ and c_{12}
  - Column 2 of Mat identifies copulas c_{23}
  - $f_{13;2}$ gives copula family of $c_{13;2}$, etc.
  - Parameter values are stored similarly as Fam matrix

- $c_{13} - c_{23} - c_{12;3}$:

$$\text{Mat:=}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix} \text{ and Fam:=}\begin{bmatrix} 0 & 0 & 0 \\ f_{12;3} & 0 & 0 \\ f_{13} & f_{23} & 0 \end{bmatrix}.$$

- $c_{12} - c_{13} - c_{23;1}$:

$$\text{Mat:=}\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 3 & 3 \end{bmatrix} \text{ and Fam:=}\begin{bmatrix} 0 & 0 & 0 \\ f_{23;1} & 0 & 0 \\ f_{12} & f_{13} & 0 \end{bmatrix}.$$

Estimation in $c_{12} - c_{23} - c_{13;2}$ (Part 1)

- **Data:** $\{(x_{i1}, x_{i2}, x_{i3}), i = 1, \cdots, n\}$ i.i.d.
- **Model:**
  - $X_j \sim F_j(\cdot, \eta_j); j = 1, 2, 3$ with η_j marginal parameter
  - $U_j := F_j(X_j, \eta_j), j = 1, 2, 3$
  - (U_1, U_2, U_3) has copula density with parameter vector $\boldsymbol{\theta} = (\theta_{12}, \theta_{23}, \theta_{13;2})$

$$
\begin{aligned}
c(u_1, u_2, u_3; \boldsymbol{\theta}) &= c_{12}(u_1, u_2, \theta_{12}) \times c_{23}(u_2, u_3; \theta_{23}) \\
&\times c_{13;2}(C_{1|2}(u_1|u_2; \theta_{12}), C_{2|3}(u_2|u_3; \theta_{23}; \theta_{13;2})
\end{aligned}
$$

- **Marginal estimation:** For each margin j estimate η_j by ML estimation to get $\hat{\eta}_j$.
- **Create pseudo copula data:** Define $\hat{u}_{ij} := F_j(x_{ij}, \hat{\eta}_j)$, then $(\hat{u}_{i1}, \hat{u}_{i2}, \hat{u}_{i3})$ is an approximate i.i.d. sample from $c(u_1, u_2, u_3; \boldsymbol{\theta})$

Estimation in $c_{12} - c_{23} - c_{13;2}$ (Part 2)

Copula parameters: $\boldsymbol{\theta} = (\theta_{12}, \theta_{23}, \theta_{13;2})$
Pseudo copula observations: $\hat{\mathbf{u}} := \{(\hat{u}_{i1}, \hat{u}_{i2}, \hat{u}_{i3}), i = 1, \cdots, n\}$

Sequential estimates:

- Estimate θ_{12} from $\{(\hat{u}_{i1}, \hat{u}_{i2}), i = 1, \cdots, n\}$
- Estimate θ_{23} from $\{(\hat{u}_{i2}, \hat{u}_{i3}), i = 1, \cdots, n\}$.
- Define pseudo observations for conditional copula

$$\hat{v}_{1|2i} := C(\hat{u}_{i1}|\hat{u}_{i2}; \hat{\theta}_{12}) \text{ and } \hat{v}_{3|2i} := C(\hat{u}_{i3}|\hat{u}_{i2}; \hat{\theta}_{23})$$

Finally estimate $\theta_{13;2}$ from $\{(\hat{v}_{1|2i}, \hat{v}_{3|2i}), i = 1, \cdots, n\}$.

Joint copula maximum likelihood

$$\begin{aligned} L(\boldsymbol{\theta}|\hat{\mathbf{u}}) &= \sum_{i=1}^{n} [\log c_{12}(\hat{u}_{i1}, \hat{u}_{i2}; \theta_{12}) + \log c_{23}(\hat{u}_{ii}, \hat{u}_{i3}; \theta_{23}) \\ &+ \log c_{13;2}(C(\hat{u}_{i1}|\hat{u}_{i2}; \theta_{12}), C(\hat{u}_{i3}|\hat{u}_{i2}; \theta_{23}); \theta_{13;2})] \end{aligned}$$

Summary

- we studied multivariate distributions
  - we identified their conditional distributions
  - we studied bivariate dependence measures
- we introduced the concept of a copula,
  - studied bivariate copula classes
  - developed graphical tools to identify copula class
  - studied estimation and model selection
- we constructed three dimensional distributions
  - with arbitrary margins and three pair copulas
  - derived a sequential estimation method for copula parameters
  - showed how the models can be stored
  - illustrated all concepts with three weight variables from the Abalone data set using VineCopula

References

Akaike, H. (1973).
Information theory and an extension of the maximum likelihood principle.
In B. N. Petrov and F. Csaki (Eds.), *Proceedings of the Second International Symposium on Information Theory Budapest*, Akademiai Kiado, pp. 267–281.

Genest, C., K. Ghoudi, and L. Rivest (1995).
A semi-parametric estimation procedure of dependence parameters in multivariate families of distributions.
Biometrika 82, 543–552.

Joe, H. (1997).
Multivariate models and multivariate dependence concepts.
CRC Press.

Joe, H. and J. Xu (1996).
The estimation method of inference functions for margins for multivariate models.
Technical Report 166, Department of Statistics, University of British Columbia.

Kojadinovic, I. and J. Yan (2010).
Modeling multivariate distributions with continuous margins using the copula R package.
Journal of Statistical Software 34(9), 1–20.

Nelsen, R. (2006).
An Introduction to Copulas.
New York: Springer.

Schepsmeier, U., J. Stöber, E. C. Brechmann, B. Gräler, T. Nagler, and T. Erhardt (2017).
VineCopula: Statistical Inference of Vine Copulas.
Version 2.1.2.

References

Scherer, M. and J.-F. Mai (2017).
Simulating copulas: stochastic models, sampling algorithms, and applications (2 ed.), Volume 6 of *Series in Quantitative Finance*.
Imperial College Press.

Schwarz, G. (1978).
Estimating the dimension of a model.
The Annals of Statistics 6(2), 461–464.

Sklar, A. (1959).
Fonctions de répartition à n dimensions et leurs marges.
Publications de l'Institut de Statistique de L'Université de Paris 8, 229–231.