Analyzing dependent data with vine copulas (Lecture 1)

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Outline

1 Motivation

2 Multivariate distributions

3 Bivariate copulas

4 Pair-copula constructions (PCC) of vine distributions (d=3)
What to expect:

There are about 1600 items found in google scholar with the search expression vine copula, over 200 of them are in 2018. A word cloud with the 20 most used words in articles in 2018 shows...
The **abalone** dataset is available from the University of California Irvine (UCI) machine learning repository http://archive.ics.uci.edu/ml/datasets/Abalone

- **Sex** / nominal / – / M, F, and I (infant)
- **Length** / continuous / mm / Longest shell measurement
- **Diameter** / continuous / mm / perpendicular to length
- **Height** / continuous / mm / with meat in shell
- **Whole weight** / continuous / grams / whole abalone
- **Shucked weight** / continuous / grams / weight of meat
- **Viscera weight** / continuous / grams / gut weight (after bleeding)
- **Shell weight** / continuous / grams / after being dried
- **Rings** / integer / – / +1.5 gives the age in years
Overview

1. Motivation
2. Multivariate distributions
3. Bivariate copulas
4. Pair-copula constructions (PCC) of vine distributions (d=3)
Outline

1 Motivation

2 Multivariate distributions

3 Bivariate copulas

4 Pair-copula constructions (PCC) of vine distributions (d=3)
Multivariate distributions

- Multivariate distributions describe stochastic behavior of several variables jointly.
- Marginal distributions describe stochastic behavior of a single variable (examples: univariate normal, exponential)

Multivariate normal \((\rho = 0, \rho = 0.8, \rho = -0.5)\)

How to construct multivariate distributions with different margins?
Bivariate normal density and contour plots

**joint density plot** (right: $\rho = 0$, middle: $\rho = .8$, left: $\rho = -.25$)

**contour plot**
Conditional distributions

- Vine distributions are defined using conditional distributions.
- Conditional distributions describe the stochastic behaviour of variables under the condition that other variables are fixed.
- Conditional = unconditional distributions if variables are independent.

- Conditional CDF of $X_i$ given $X_j = x_j$:

$$F_{i|j}(x_i|x_j) = \frac{1}{f_j(x_j)} \frac{\partial}{\partial x_j} F_{ij}(x_i, x_j)$$

- Conditional PDF of $(X_i, X_j)$ given that $X_k = x_k$:

$$f_{i,j|k}(x_i, x_j|x_k) := \frac{f_{ijk}(x_i, x_j, x_k)}{f_k(x_k)}$$
Multivariate distributions

Weight variables in Abalone data

- shuck
- vis
- shell
Most well known dependency measure is the correlation $\rho$ between two random variables.

It only measures linear dependencies.

Non linear dependencies can be detected by Kendall’s $\tau$ which measures the difference between the concordance and discordance probability.

Upper (lower) tail dependence measures the probability of joint large (small) occurrences as one moves to the extremes.

Multivariate normal has no tail dependence, while the multivariate t distribution has tail dependence.

When upper and lower tail dependence are not the same we speak of asymmetric tail dependence.

How to separate dependency patterns from the marginal behavior?
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Resources for copulas

- Copula theory started with Sklar (1959)
- **Books on copulas:**
  - Multivariate models and multivariate dependence concepts by (Joe 1997)
  - An Introduction to Copulas (Nelsen 2006)
  - Simulating copulas: stochastic models, sampling algorithms, and applications (2nd edition) (Scherer and Mai 2017)
- **Software:**
  - copula (Kojadinovic and Yan 2010)
  - VineCopula (Schepsmeier et al. 2017)
## Copula approach

Consider $d$ random variables $\mathbf{X} = (X_1, \ldots, X_d)$ with

<table>
<thead>
<tr>
<th>pdf</th>
<th>cdf</th>
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</thead>
<tbody>
<tr>
<td>$f_i(x_i)$, $i = 1, \ldots, d$</td>
<td>$F_i(x_i)$, $i = 1, \ldots, nd$</td>
</tr>
</tbody>
</table>

### Marginal

| $f(x_1, \ldots, x_d)$ | $F(x_1, \ldots, x_d)$ |

### Conditional

| $f(\cdot | \cdot)$ | $F(\cdot | \cdot)$ |

### Copula (distribution)

A copula $C(u_1, \ldots, u_d)$ is a multivariate distribution on $[0, 1]^d$ with uniformly distributed marginals.

### Sklar’s theorem (1959)

A joint distribution function $F$ with margins $F_j, j = 1, \ldots, d$ can be expressed as

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \tag{1}$$

for some copula $C$. It is unique when $F$ is absolutely continuous.
Sklar’s theorem for (conditional) densities

Densities and conditional densities for \( d=2 \)

Let \( f_{12} \) denote the density of the bivariate distribution \( F_{12} \), then

\[
\begin{align*}
f_{12}(x_1, x_2) &= c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) \cdot f_2(x_2) \\
f_{2|1}(x_2|x_1) &= c_{12}(F_1(x_1), F_2(x_2)) \cdot f_2(x_2)
\end{align*}
\]

(2)

where \( c_{12}(\cdot) \) is the density associated with the copula \( C_{12} \).

Equations (1) and (2) can also be used in a constructive way to build new multivariate distributions.
Bivariate copulas

Variable scales

Let \((X_1, \ldots, X_d) \sim F\).

- **x-scale**: original scale of variables \(X_j, j = 1, \ldots, d\)
- **u-scale**: copula scale of variables
  \[ U_j := F_j(X_j) \sim U(0, 1), j = 1, \ldots, d \]
- **z-scale**: normalized score scale of variables
  \[ Z_j := \Phi^{-1}(U_j) \sim N(0, 1), j = 1, \ldots, d \]

For \(d = 2\) pair plots and density contour plots on

- **x-scale** mix dependence and marginal effects
- **u-scale** show dependence effects, but not so informative
- **z scale** show dependence effects and can be compared to known behavior of bivariate normal variables. The associated contour plot is called a **normalized contour plot**
**Bivariate copula families**

- **Elliptical copulas:** Copula constructed using inversion of Sklar’s theorem applied to bivariate elliptical distributions such as the bivariate normal or Student t distribution.

- **Archimedean copulas:** Copulas directly constructed using a strictly monotone convex generator function $\psi$ with $\psi(0) = 0$

$$C(u_1, u_2) = \psi^{-1}[\psi(u_1) + \psi(u_2)], \quad (3)$$

where $\psi^{-1}$ denotes the pseudo inverse of $\psi$. Examples are Gumbel, Clayton and Frank copulas.

- **Extreme value copulas:** This class of copulas associated with limiting distributions of bivariate extreme value theory. An interesting nonsymmetric class is the Tawn copula with 3 parameters.
Bivariate elliptical copula families

**Gaussian copula**
(left $\tau = .25$, right: $\tau = .75$)

**t-copula with $df = 3$**
(left $\tau = .25$, right: $\tau = .75$)
Bivariate Archimedean copula families

**Gumbel copula**
(left $\tau = 0.25$, right: $\tau = 0.75$)

**Clayton copula**
(left $\tau = 0.25$, right: $\tau = 0.75$)
Bivariate copulas

Bivariate Tawn copula with 2 parameters

\[ \theta = 2, \psi_1 = 0.8 \]

\[ \theta = 5, \psi_1 = 0.8 \]

\[ \theta = 5, \psi_1 = 0.2 \]
Some formulas of bivariate copulas (part 1)

**Gaussian copula:**

\[ C(u_1, u_2; \rho) = \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho), \]

where \( \Phi(\cdot) \) is \( N(0, 1) \) cdf and \( \Phi_2(\cdot, \cdot; \rho) \) is bivariate normal cdf with zero means, unit variances and correlation \( \rho \). The pdf is

\[ c(u_1, u_2; \rho) = \frac{1}{\phi(x_1)\phi(x_2)} \frac{1}{\sqrt{1 - \rho^2}} \exp \left\{ -\frac{\rho^2(x_1^2 + x_2^2) - 2\rho x_1 x_2}{2(1 - \rho^2)} \right\}, \]

where \( x_1 := \Phi^{-1}(u_1) \) and \( x_2 := \Phi^{-1}(u_2) \).

**Student t copula:**

\[ c(u_1, u_2; \nu, \rho) = \frac{t(T_{\nu}^{-1}(v_1), T_{\nu}^{-1}(v_2); \nu, \rho)}{t_{\nu}(T_{\nu}^{-1}(v_1))t_{\nu}(T_{\nu}^{-1}(v_2))}, \]

where \( T_{\nu}(t_{\nu}) \) are univariate Student t cdf (pdf) with \( df = \nu \) and \( t(\cdot, \cdot; \nu, \rho) \) pdf of bivariate Student t with \( df = \nu \) and scale matrix \( \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \).
Some formulas of bivariate copulas (part 2)

- **Clayton copula:**
  
  \[ C(u_1, u_2) = (u_1^{-\delta} + u_2^{-\delta} - 1)^{-\frac{1}{\delta}}, \]

  where \( 0 < \delta < \infty \), \( \delta \to 0 \) corresponds to independence

- **Gumbel copula:**
  
  \[ C(u_1, u_2) = \exp[-\{(\ln u_1)^\delta + (\ln u_2)^\delta\}^{\frac{1}{\delta}}], \]

  where \( \delta \geq 1 \), \( \delta = 1 \) corresponds to independence.
Kendall’s $\tau$ of some copula families

Kendall’s tau

The Kendall’s $\tau$ between $X_1$ and $X_2$ is defined as

$$\tau := P((X_{11} - X_{21})(X_{12} - X_{22}) > 0) - P((X_{11} - X_{21})(X_{12} - X_{22}) < 0),$$

where $(X_{11}, X_{12})$ and $(X_{21}, X_{22})$ are i.i.d copies of $(X_1, X_2)$. Further for the associated copula $C$ we can express

$$\tau = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2).$$

<table>
<thead>
<tr>
<th>Family</th>
<th>Kendall’s $\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>$\tau = \frac{2}{\pi} \arcsin(\rho)$</td>
</tr>
<tr>
<td>Student t</td>
<td>$\tau = \frac{2}{\pi} \arcsin(\rho)$</td>
</tr>
<tr>
<td>Clayton</td>
<td>$\tau = \frac{\delta}{\delta + 2}$</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$\tau = 1 - \frac{1}{\delta}$</td>
</tr>
</tbody>
</table>
Bivariate tail dependence

Upper and lower tail dependence coefficient

\[ \lambda^{upper} = \lim_{t \to 1^-} P(X_2 > F_2^{-1}(t)|X_1 > F_1^{-1}(t)) = \lim_{t \to 1^-} \frac{1 - 2t + C(t, t)}{1 - t}, \]

\[ \lambda^{lower} = \lim_{t \to 0^+} P(X_2 < F_2^{-1}(t)|X_1 < F_1^{-1}(t)) = \lim_{t \to 0^+} \frac{C(t, t)}{t}. \]

Illustration: upper tail (red), lower tail (blue)
### Bivariate copulas

#### Tail dependence of bivariate copula families

<table>
<thead>
<tr>
<th>Family</th>
<th>Upper tail</th>
<th>Lower tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>$\lambda^{up} = 0$</td>
<td>$\lambda^{low} = 0$</td>
</tr>
<tr>
<td>Student t</td>
<td>$\lambda^{up} = 2t_{\nu+1} \left( -\sqrt{\nu + 1} \frac{1-\rho}{1+\rho} \right)$</td>
<td>$\lambda^{low} = 2t_{\nu+1} \left( -\sqrt{\nu + 1} \frac{1-\rho}{1+\rho} \right)$</td>
</tr>
<tr>
<td>Clayton</td>
<td>$\lambda^{up} = 0$</td>
<td>$\lambda^{low} = 2^{-1/\delta}$</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$\lambda^{up} = 2 - 2^{1/\delta}$</td>
<td>$\lambda^{low} = 0$</td>
</tr>
</tbody>
</table>
Meta distributions

are build using a copula \([u_1, u_2]\) and different margins (normal/exponential \([x_1, x_2]\) or normal/normal \([z_1, z_2]\))

Gaussian copula

Clayton copula
Bivariate rotations

- **Extension:** To extend range of dependence we use counterclockwise rotations
  - 90 degree: \( c_{90}(u_1, u_2) := c(1 - u_2, u_2) \)
  - 180 degree: \( c_{180}(u_1, u_2) := c(1 - u_1, 1 - u_2) \)
  - 270 degree: \( c_{270}(u_1, u_2) := c(u_2, 1 - u_1) \)

- **Extended Clayton:**
  \[
  c_{\text{clayton}}^{\text{extended}}(u_1, u_2; \delta) := \begin{cases} 
    c_{\text{clayton}}(u_1, u_2) & \text{if } \delta > 0 \\
    c_{\text{clayton}}(1 - u_2, u_1) & \text{otherwise}
  \end{cases}
  \]

- **Exchangeability or reflection symmetry:**
  \[
  c(u_1, u_2) = c(u_2, u_1) \text{ for all } u_1, u_2
  \]
  - Gumbel and Clayton are exchangeable
  - 90 or 270 degree rotation is no longer exchangeable
Illustration of rotations

Figure: Normalized contour plots of Clayton rotations: top left: 0 degree rotation ($\tau = .5$), top right: 90 degree rotation ($\tau = -.5$), bottom left: 180 degree rotation ($\tau = .5$), bottom right: 270 degree rotation ($\tau = -.5$).
Bivariate copulas

Parametric bivariate copula models

- **Data:** i.i.d observations \((x_{i1}, x_{i2}), i = 1, \cdots, n\) from the joint density \(f_{12}(x_1, x_2) = c_{12}(F_1(x_1), F_2(x_2))f_1(x_1)f_2(x_2)\).

- **Margins:** \(F_j(x_j; \theta_{mj}), j = 1, 2\) with marginal parameters \(\theta_m = (\theta_{m1}, \theta_{m2})\).

- **Copula:** \(c_{12}(u_1, u_2; \theta_c)\) with copula parameter \(\theta_c\).

- **Estimation:**
  - **Joint:** Marginal and copula parameter are jointly estimated using maximum likelihood (ML).
  - **Two step:**
    - **Inference for margins:** Estimate margin separately to get \(\hat{F}_{j\text{par}}\) and then use ML based on \(\hat{u}_{ij} = \hat{F}_{j\text{par}}(x_{ij})\). Joe and Xu (1996)
    - **Semiparametric approach:** Estimate margins using empirical cdf’s \(\hat{F}_j\) and then use ML based on \(\hat{u}_{ij} = \hat{F}_j(x_{ij})\). Genest et al. (1995)
Nonparametric bivariate copula models

- **Data:** i.i.d observations \((x_{i1}, x_{i2}), i = 1, \cdots, n\) from the joint density \(f_{12}(x_1, x_2) = c_{12}(F_1(x_1), F_2(x_2)) f_1(x_1)f_2(x_2)\).

- **Models:** Both marginal and copula models are not specified.

- **Two step estimation:**
  - Margins are estimated using empirical cdf’s \(\hat{F}_j\). The empirical copula is estimated based \(\hat{u}_{ij} = \hat{F}_j(x_{ij})\).
  - Margins are estimated using kernel density cdf estimates \(\hat{F}_j^{kd}\) and copula density is estimated by bivariate kernel estimates based on \(\hat{u}_{ij}^{ks} = \hat{F}_j^{ks}(x_{ij})\).
Let \( \ell_n(\hat{\theta}, x) \) be the log likelihood based on model with \( p \) dimensional parameter \( \theta \) and observed data \( x \) of size \( n \) evaluated at the estimate \( \hat{\theta} \).

**AIC: (Akaike 1973)**

\[
AIC_n := 2\ell_n(\hat{\theta}, x) + 2p
\]

**BIC: (Schwarz 1978)**

\[
BIC_n := 2\ell_n(\hat{\theta}, x) + \log(n)p
\]
## Bivariate copula estimation in VineCopula

The R package **VineCopula** allows also for bivariate copula estimation.

<table>
<thead>
<tr>
<th>Function</th>
<th>Purpose</th>
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<td>Constructing BiCop-objects</td>
</tr>
<tr>
<td>BiCopCDF</td>
<td>Distribution Function of a Bivariate Copula</td>
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<tr>
<td>BiCopCondSim</td>
<td>Conditional simulation from a Bivariate Copula</td>
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<td>BiCopEst</td>
<td>Parameter Estimation for Bivariate Copula Data</td>
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<td>BiCopEstList</td>
<td>List of Maximum Likelihood Estimates for Several Bivariate Copula Families</td>
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<td>BiCopGofTest</td>
<td>Goodness-of-Fit Test for Bivariate Copulas</td>
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<td>BiCopHfunc</td>
<td>Conditional Distribution Function of a Bivariate Copula</td>
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<td>Kernel estimate of a Bivariate Copula Density</td>
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<td>BiCopMetaContour</td>
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<td>BiCopPar2TailDepTail</td>
<td>Dependence Coefficients of a Bivariate Copula</td>
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<td>Kendall’s Tau Value of a Bivariate Copula</td>
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<td>BiCopSim</td>
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<tr>
<td>BiCopTau2Par</td>
<td>Parameter of a Bivariate Copula for a given Kendall’s Tau Value</td>
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</tbody>
</table>
Outline

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Motivation for vine based models

- Many data structures exhibit
  - different marginal distributions
  - non-symmetric dependencies between some pairs of variables
  - heavy tail dependencies between some pairs of variables

- These cannot be modeled with a Gaussian or multivariate t distribution

- The copula approach allows to model dependencies and marginal distributions separately.

- Marginal time dependencies can be captured by appropriate univariate time series models.

- Elliptical and Archimedean copulas do not allow for different dependency patterns between pairs of variables.

Vine based models can overcome all these shortcomings.
Some (notational) remarks

- We distinguish between the copula associated with a bivariate conditional distribution and a bivariate conditional distribution derived from the copula variables. In particular
  - The conditional distribution of \((X_i, X_j)\) given \(X_D = x_d\) has copula \(C_{ij;D}(\cdot, \cdot)\). We call \(C_{ij;D}(\cdot, \cdot)\) a conditional copula.
  - Assuming that \((U_1, \ldots, U_d)\) have the copula \(C\) as distribution function, the bivariate distribution of \((U_i, U_j)\) given \(U_d = u_d\) is denoted by \(C_{ij|D}(\cdot, \cdot)\). This is in general not a copula.

- Specification of three bivariate copulas does not lead in general to a valid construction of three variate copula.
Pair-copula constructions in 3 dimensions

\[ f(x_1, x_2, x_3) = f_{3|12}(x_3|x_1, x_2)f_{2|1}(x_2|x_1)f_1(x_1) \]

Using Sklar for \( f(x_1, x_2), f(x_2, x_3) \) and \( f_{13|2}(x_1, x_3|x_2) \) implies

\[ f_{2|1}(x_2|x_1) = c_{12}(F_1(x_1), F_2(x_2))f_2(x_2) \]
\[ f_{3|12}(x_3|x_1, x_2) = c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))f_{3|2}(x_3|x_2) \]
\[ = c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))c_{23}(F_2(x_2), F_3(x_3))f_3(x_3) \]

\[ f(x_1, x_2, x_3) = c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))c_{23}(F_2(x_2), F_3(x_3)) \]
\[ \times c_{12}(F_1(x_1), F_2(x_2)) \]
\[ \times f_3(x_3)f_2(x_2)f_1(x_1) \]

Only bivariate copulas and univariate conditional cdf's are used. We will later generalize this to \( d \) dimensions.
The bivariate copulas in occurring in a PCC are called pair copulas.

Pair copulas can be parametrically modelled with parameter $\theta$, i.e. we write $C_{ij}(\cdot, \cdot; \theta)$.

The conditional copulas $C_{ij;D}$ depend generally on the conditioning value $x_D$, we therefore use $C_{ij;D}(\cdot, \cdot; x_D)$.

**Simplifying assumption**

If there is no dependency, i.e.

$$C_{ij;D}(\cdot, \cdot; x_D) = C_{ij;D}(\cdot, \cdot)$$

for all $x_D$, we say that the **simplifying assumption** holds.
In the PCC we can reorder the variables, therefore we get three PCC’s.

<table>
<thead>
<tr>
<th>Three simplified PCC’s in 3 dimensions</th>
</tr>
</thead>
</table>
| \( \mathbf{c}_{12} - \mathbf{c}_{23} - \mathbf{c}_{13;2} \) : | \[
  f(x_1, x_2, x_3) = c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))c_{12}(F_1(x_1), F_2(x_2)) \\
  \times c_{23}(F_2(x_2), F_3(x_3))f_1(x_1)f_2(x_2)f_3(x_3)
\] |
| \( \mathbf{c}_{13} - \mathbf{c}_{23} - \mathbf{c}_{12;3} \) : | \[
  f(x_1, x_2, x_3) = c_{12;3}(F_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3))c_{13}(F_1(x_1), F_3(x_3)) \\
  \times c_{23}(F_2(x_2), F_3(x_3))f_1(x_1)f_2(x_2)f_3(x_3)
\] |
| \( \mathbf{c}_{12} - \mathbf{c}_{13} - \mathbf{c}_{23;1} \) : | \[
  f(x_1, x_2, x_3) = c_{23;1}(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1))c_{12}(F_1(x_1), F_2(x_2)) \\
  \times c_{13}(F_1(x_1), F_3(x_3))f_1(x_1)f_2(x_2)f_3(x_3)
\] |
Storing the PCC with matrices

- \( c_{12} - c_{23} - c_{13;2} : \)
  \[
  \begin{bmatrix}
  1 & 0 & 0 \\
  3 & 2 & 0 \\
  2 & 3 & 3 \\
  \end{bmatrix}
  \]
  and \( \begin{bmatrix}
  0 & 0 & 0 \\
  f_{13;2} & 0 & 0 \\
  f_{13} & f_{23} & 0 \\
  \end{bmatrix} \).
  - Column 1 of Mat identifies copulas \( c_{13;2} \) and \( c_{12} \)
  - Column 2 of Mat identifies copulas \( c_{23} \)
  - \( f_{13;2} \) gives copula family of \( c_{13;2} \), etc.
  - Parameter values are stored similarly as Fam matrix

- \( c_{13} - c_{23} - c_{12;3} : \)
  \[
  \begin{bmatrix}
  1 & 0 & 0 \\
  2 & 2 & 0 \\
  3 & 3 & 3 \\
  \end{bmatrix}
  \]
  and \( \begin{bmatrix}
  0 & 0 & 0 \\
  f_{12;3} & 0 & 0 \\
  f_{13} & f_{23} & 0 \\
  \end{bmatrix} \).

- \( c_{12} - c_{13} - c_{23;1} : \)
  \[
  \begin{bmatrix}
  2 & 0 & 0 \\
  3 & 1 & 0 \\
  1 & 3 & 3 \\
  \end{bmatrix}
  \]
  and \( \begin{bmatrix}
  0 & 0 & 0 \\
  f_{23;1} & 0 & 0 \\
  f_{12} & f_{13} & 0 \\
  \end{bmatrix} \).
**Estimation in** $c_{12} - c_{23} - c_{13;2}$ (Part 1)

- **Data:** $\{(x_{i1}, x_{i2}, x_{i3}), i = 1, \cdots, n\}$ i.i.d.

- **Model:**
  - $X_j \sim F_j(\cdot, \eta_j); j = 1, 2, 3$ with $\eta_j$ marginal parameter
  - $U_j := F_j(X_j, \eta_j), j = 1, 2, 3$
  - $(U_1, U_2, U_3)$ has copula density with parameter vector $\theta = (\theta_{12}, \theta_{23}, \theta_{13;2})$

\[
c(u_1, u_2, u_3; \theta) = c_{12}(u_1, u_2, \theta_{12}) \times c_{23}(u_2, u_3; \theta_{23}) \times c_{13;2}(C_1|2(u_1|u_2; \theta_{12}), C_2|3(u_2|u_3; \theta_{23}; \theta_{13;2})
\]

- **Marginal estimation:** For each margin $j$ estimate $\eta_j$ by ML estimation to get $\hat{\eta}_j$.

- **Create pseudo copula data:** Define $\hat{u}_{ij} := F_j(x_{ij}, \hat{\eta}_j)$, then $(\hat{u}_{i1}, \hat{u}_{i2}, \hat{u}_{i3})$ is an approximate i.i.d. sample from $c(u_1, u_2, u_3; \theta)$
Estimation in $c_{12} - c_{23} - c_{13;2}$ (Part 2)

Copula parameters: $\theta = (\theta_{12}, \theta_{23}, \theta_{13;2})$

Pseudo copula observations: $\hat{\mathbf{u}} := \{ (\hat{u}_{i1}, \hat{u}_{i2}, \hat{u}_{i3}), i = 1, \ldots, n \}$

Sequential estimates:

- Estimate $\theta_{12}$ from $\{ (\hat{u}_{i1}, \hat{u}_{i2}), i = 1, \ldots, n \}$
- Estimate $\theta_{23}$ from $\{ (\hat{u}_{i2}, \hat{u}_{i3}), i = 1, \ldots, n \}$
- Define pseudo observations for conditional copula $\hat{v}_{1|2i} := C(\hat{u}_{i1}|\hat{u}_{i2}; \hat{\theta}_{12})$ and $\hat{v}_{3|2i} := C(\hat{u}_{i3}|\hat{u}_{i2}; \hat{\theta}_{23})$

Finally estimate $\theta_{13;2}$ from $\{ (\hat{v}_{1|2i}, \hat{v}_{3|2i}), i = 1, \ldots, n \}$.

Joint copula maximum likelihood

$$L(\theta|\hat{\mathbf{u}}) = \sum_{i=1}^{n} \left[ \log c_{12}(\hat{u}_{i1}, \hat{u}_{i2}; \theta_{12}) + \log c_{23}(\hat{u}_{ij}, \hat{u}_{i3}; \theta_{23}) ight]$$
$$+ \log c_{13;2}(C(\hat{u}_{i1}|\hat{u}_{i2}; \theta_{12}), C(\hat{u}_{i3}|\hat{u}_{i2}; \theta_{23}); \theta_{13;2})$$
Summary

- We studied multivariate distributions
  - We identified their conditional distributions
  - We studied bivariate dependence measures

- We introduced the concept of a copula,
  - Studied bivariate copula classes
  - Developed graphical tools to identify copula class
  - Studied estimation and model selection

- We constructed three dimensional distributions
  - With arbitrary margins and three pair copulas
  - Derived a sequential estimation method for copula parameters
  - Showed how the models can be stored
  - Illustrated all concepts with three weight variables from the Abalone data set using VineCopula
References


*Simulating copulas: stochastic models, sampling algorithms, and applications* (2 ed.), Volume 6 of *Series in Quantitative Finance*.  
Imperial College Press.

Estimating the dimension of a model.  

Fonctions de répartition à n dimensions et leurs marges.  
*Publications de l’Institut de Statistique de L’Université de Paris* 8, 229–231.