Empirical Processes with Applications in Statistics

3. Glivenko–Cantelli and Donsker Theorems

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Set-up

- i.i.d. random variables $X_1, X_2, \ldots$ on some probability space Ω taking values in some measurable space $(\mathcal{X}, \mathcal{A})$ with common law P

$$\mathrm{P}(X_i \in B) = P(B), \qquad \forall B \in \mathcal{A}$$

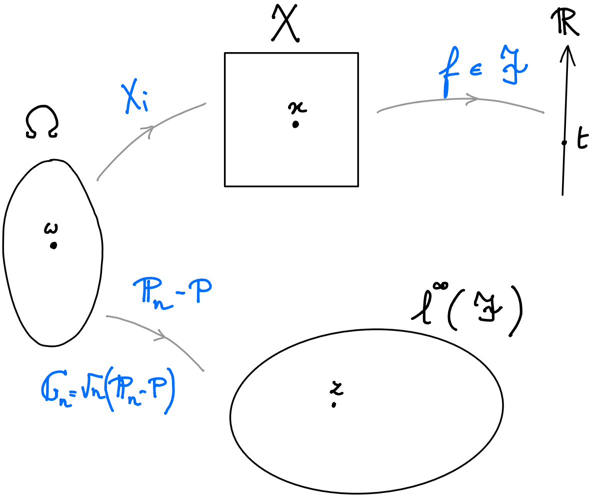
- Family $\mathcal{F} \subset L_1(P)$ of P-integrable functions $f : \mathcal{X} \to \mathbb{R}$
- *Empirical and population measures*: for $f \in \mathcal{F}$,

$$\mathbb{P}_n f = \frac{1}{n} \sum_{i=1}^{n} f(X_i)$$

$$Pf = \int_{\mathcal{X}} f(x)\, dP(x) = \mathrm{E}[f(X_i)]$$

- *Empirical process*:

$$\mathbb{G}_n = \sqrt{n}\,(\mathbb{P}_n - P)$$


$\mathcal{X}$
$\mathbb{R}$
X_i
x
$f \in \mathcal{F}$
t
Ω
ω
$\mathbb{P}_n - P$
$\ell^\infty(\mathcal{F})$
$\mathbb{G}_n = \sqrt{n}(\mathbb{P}_n - P)$
z

Bounded stochastic processes

Assume that the following map is bounded almost surely:

$$\mathcal{F} \to \mathbb{R} : f \mapsto \mathbb{P}_n f - Pf$$

Then we can view $\mathbb{P}_n - P$ and $\mathbb{G}_n$ as (possibly non-measurable) maps

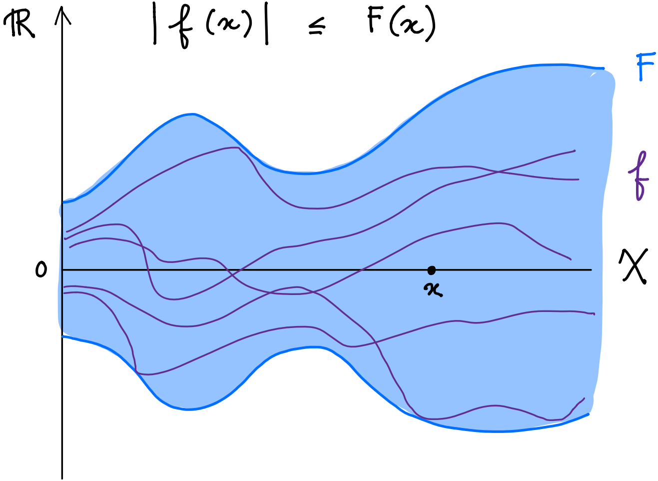
$$\Omega \to \ell^\infty(\mathcal{F})$$

- Assume that $\sup_{f \in \mathcal{F}} |f(x)| < \infty$ for P-almost every $x \in \mathcal{X}$ and $\sup_{f \in \mathcal{F}} |Pf| < \infty$
- Often, we will assume that $\mathcal{F}$ has a P-integrable *envelope* $F : \mathcal{X} \to [0, \infty)$, i.e.,

$$\forall x \in \mathcal{X}, \forall f \in \mathcal{F}, \qquad |f(x)| \leqslant F(x)$$

for all $x \in X$ and $f \in \mathcal{F}$:

$$|f(x)| \leq F(x)$$



Uniform law of large numbers: Glivenko–Cantelli

Strong law of large numbers. For all $f \in \mathcal{F}$,

$$\mathbb{P}_n f \to Pf \quad \text{a.s.}, \quad n \to \infty$$

Trivially, for every $f_1, \ldots, f_k \in \mathcal{F}$,

$$\max_{j=1,\ldots,k} |\mathbb{P}_n f - Pf| \to 0 \quad \text{a.s.}, \quad n \to \infty$$

Can we make this statement uniform in $\mathcal{F}$?

Definition: Glivenko–Cantelli. $\mathcal{F}$ is a *P-Glivenko–Cantelli class* if

$$\sup_{f \in \mathcal{F}} |\mathbb{P}_n f - Pf| \overset{\text{as}*}{\to} 0, \qquad n \to \infty$$

Uniform central limit theorem: Donsker

Multivariate central limit theorem
Finite-dimensional distributions of $\mathbb{G}_n$ converge to those of a centered Gaussian process $f \mapsto \mathbb{G}f$ on $\mathcal{F}$ with covariance function

$$E[\mathbb{G}f_1 \, \mathbb{G}f_2] = Pf_1 f_2 - Pf_1 \, Pf_2 = \operatorname{cov}(f_1(X), f_2(X))$$

for $f_1, f_2 \in \mathcal{F}$, where $X \sim P$.

Can we make this statement uniform in $f \in \mathcal{F}$?

Definition: Donsker class.
$\mathcal{F}$ is a *P-Donsker class* if $\mathbb{G}_n \rightsquigarrow \mathbb{G}$ as $n \to \infty$ in $\ell^\infty(\mathcal{F})$.

To show: *asymptotic tightness*

A counterexample

Let $\mathcal{X} = [0, 1]$ with the Borel σ-field and P the uniform distribution. Consider

$$\mathcal{F} = \{\text{all continuous functions } f : \mathcal{X} \to [0, 1]\}$$

For any $x_1, \ldots, x_n \in \mathcal{X}$ and any $\varepsilon > 0$, we can find $f \in \mathcal{F}$ such that

- $f(x_1) = \ldots = f(x_n) = 1$
- $Pf = \int_0^1 f(x)\, dx \leqslant \varepsilon$.

As a consequence,

$$\sup_{f \in \mathcal{F}} |\mathbb{P}_n f - Pf| = 1$$

The class $\mathcal{F}$ is therefore *not* P-Glivenko–Cantelli nor P-Donsker.

By the Stone–Weierstrass theorem, the example extends to the subfamily

$$\mathcal{F}_0 = \{\text{all polynomials } f : \mathcal{X} \to [0, 1] \text{ with rational coefficients}\}$$

See lecture 1.

Complexity of a class of functions

In general, how to put conditions on $\mathcal{F}$ limiting its complexity?

The previous counterexample shows that it is *not* sufficient to impose that

- functions f in $\mathcal{F}$ are bounded
- functions f in $\mathcal{F}$ are smooth
- the cardinality of $\mathcal{F}$ is not too large
- ...

Two techniques: controlling the

- *bracketing* numbers
- *covering* numbers

These will provide sufficient but not necessary conditions

Main sources for this lecture: van der Vaart and Wellner (1996), van der Vaart (1998)

Glivenko–Cantelli and Donsker Theorems

Bracketing entropy

Uniform entropy

VC-classes

Extension: Changing function classes

Definition: Bracketing numbers.

- *Bracket* induced by functions $l, u : \mathcal{X} \to \mathbb{R}$:

$$[l, u] = \{f : \mathcal{X} \to \mathbb{R} \mid \forall x \in \mathcal{X} : l(x) \leqslant f(x) \leqslant u(x)\}$$

An ε-bracket $[l, u]$: if $\|u - l\| \leqslant \varepsilon$, with $\|\cdot\|$ some norm

Norm is usually $L_1(P)$ or $L_2(P)$

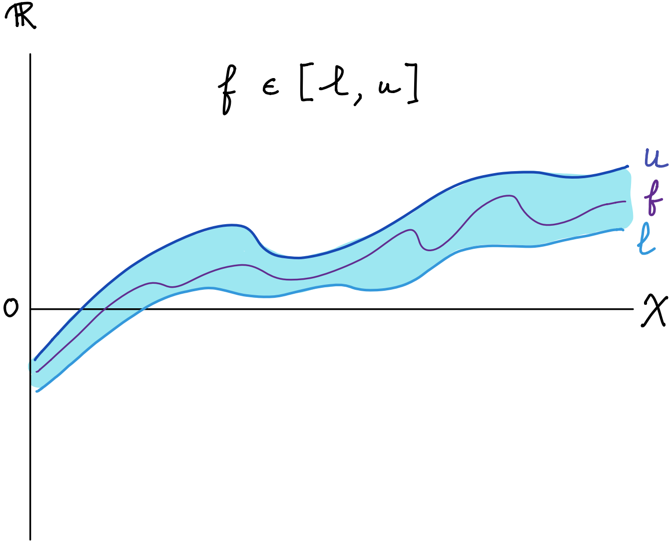
- *Bracketing number*:

$$N_{[]}(\varepsilon, \mathcal{F}, \|\cdot\|) = \text{the minimum number, } N\text{, of } \varepsilon\text{-brackets needed to cover } \mathcal{F}\text{, i.e.,} \quad \mathcal{F} \subset \bigcup_{i=1}^N [l_i, u_i] \text{ and } \|u_i - l_i\| \leqslant \varepsilon \text{ for all } i$$

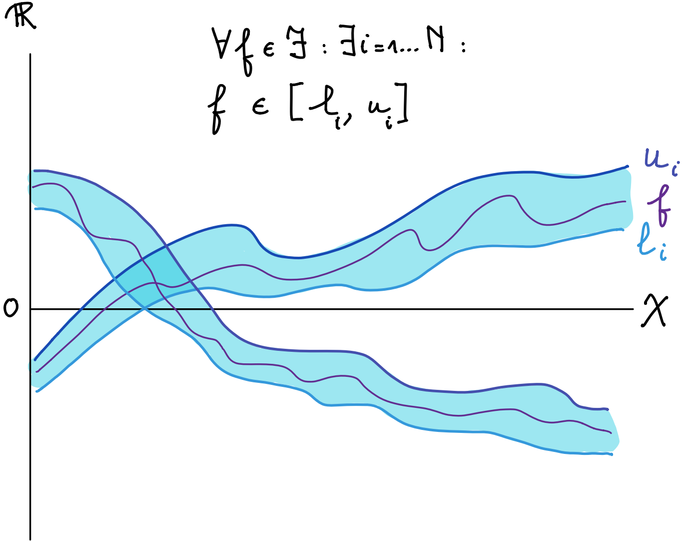
l_i and u_i need not belong to $\mathcal{F}$, but $\|u_i\|$ and $\|l_i\|$ need to be finite.

- *Entropy with bracketing*:

$$\log N_{[]}(\varepsilon, \mathcal{F}, \|\cdot\|)$$


ℝ
f ∈ [ℓ, u]
u
f
ℓ
0
X

$$\forall f \in \mathcal{F} : \exists i = 1 \dots N : f \in [l_i, u_i]$$



Glivenko–Cantelli with bracketing

- i.i.d. random elements $X_1, X_2, \ldots$ in $(\mathcal{X}, \mathcal{A})$ with common law P
- Family $\mathcal{F} \subset L_1(P)$ of P-integrable functions $f : \mathcal{X} \to \mathbb{R}$
- Law of large numbers:

$$\forall f \in \mathcal{F}, \qquad \mathbb{P}_n f \to Pf \text{ a.s.}, \quad n \to \infty$$

- Uniformly in $f \in \mathcal{F}$?

Glivenko–Cantelli theorem: bracketing. If

$$\forall \varepsilon > 0, \qquad N_{[]}(\varepsilon, \mathcal{F}, L_1(P)) < \infty$$

then $\mathcal{F}$ is P-Glivenko–Cantelli, i.e.,

$$\sup_{f \in \mathcal{F}} |\mathbb{P}_n f - Pf| \overset{\text{as}*}{\to} 0, \qquad n \to \infty.$$

Proof.

Fix $\varepsilon > 0$.

- Consider a covering $\mathcal{F} \subset \bigcup_{i=1}^{N} [l_i, u_i] \subset L_1(P)$ with ε-brackets:

$$\forall i = 1, \ldots, N, \qquad 0 \leqslant P(u_i - l_i) \leqslant \varepsilon$$

- For every $f \in \mathcal{F}$, find $i \in \{1, \ldots, N\}$ such that $f \in [l_i, u_i]$ and then

$$\begin{array}{ccccccc} P_n l_i & \leqslant & P_n f & \leqslant & P_n u_i & & \\ P l_i & \leqslant & P f & \leqslant & P u_i & \leqslant & P l_i + \varepsilon \end{array}$$

- Bound the supremum over $f \in \mathcal{F}$ by a maximum over $i = 1, \ldots, N$:

$$\sup_{f \in \mathcal{F}} |\mathbb{P}_n f - Pf| \leqslant \max_{i=1,\ldots,N} |\mathbb{P}_n l_i - P l_i| \vee |\mathbb{P}_n u_i - P u_i| + \varepsilon$$

- Apply the law of large numbers to $l_1, \ldots, l_N$ and $u_1, \ldots, u_N$.

Let $\varepsilon \downarrow 0$. □

Donsker theorem with bracketing

Recall $\mathbb{G}_n = \sqrt{n}(\mathbb{P}_n - P)$ seen as map $\Omega \to \ell^\infty(\mathcal{F})$, for $\mathcal{F} \subset L_2(P)$

Donsker theorem: bracketing.
If for some (and then for all) $\delta > 0$ we have a finite *bracketing integral*

$$J_{[]}(\delta, \mathcal{F}, L_2(P)) := \int_0^\delta \sqrt{\log N_{[]}(\varepsilon, \mathcal{F}, L_2(P))}\, d\varepsilon < \infty$$

then $\mathcal{F}$ is P-Donsker, i.e.,

$$\mathbb{G}_n \rightsquigarrow \mathbb{G}, \qquad n \to \infty, \qquad \text{in } \ell^\infty(\mathcal{F})$$

- $L_2(P)$-brackets $[l_i, u_i]$ satisfy $P[(u_i - l_i)^2] \leqslant \varepsilon^2$ and thus $P(u_i - l_i) \leqslant \varepsilon$
  $\Longrightarrow$ $L_2(P)$ brackets are *smaller* than $L_1(P)$ brackets
  $\Longrightarrow$ *higher* bracketing number $N_{[]}(\varepsilon, \mathcal{F}, L_2(P))$
- Finite bracketing integral
  $\Longrightarrow$ $N_{[]}(\varepsilon, \mathcal{F}, L_2(P))$ cannot go to ∞ too quickly as $\varepsilon \downarrow 0$

Proof.

Show *asymptotic tightness* of $\mathbb{G}_n$ via finite-partition criterion (part 2, p. 20).

Construct partition $\mathcal{F} = \bigcup_{i=1}^{N} \mathcal{F}_i$ from δ-brackets $[l_i, u_i]$.

Use *maximal inequality*: there exists finite $a(\delta) > 0$ such that

$$\mathrm{E}^* \left[\max_{i=1,\dots,N} \sup_{f,g \in \mathcal{F}_i} |\mathbb{G}_n f - \mathbb{G}_n g| \right] \lesssim J_{[]}(\delta, \mathcal{F}, L_2(P)) + a(\delta)^{-1} P \left[F^2 \, \mathbb{1}\left\{ F > a(\delta) \sqrt{n} \right\} \right]$$

with $F \in L_2(P)$ an envelope of $\mathcal{F}$, constructed via 1-brackets.

Let first $n \to \infty$, then $\delta \to 0$, and apply Markov's inequality. □

Proof of the maximal inequality is difficult. Techniques: Bernstein's inequality, Orlicz norms, chaining. See Lemmas 19.32–34 in van der Vaart (1998).

Example: weighted distribution function

Suppose $\mathcal{X} = [0, 1]$ and P the uniform distribution.

Weighted empirical distribution function: for $w : [0, 1] \to [0, \infty]$, let

$$
\begin{aligned}
f_t(x) &= w(t)\mathbb{1}_{[0,t]}(x), \qquad x, t \in [0, 1] \\
\mathbb{G}_n f_t &= \sqrt{n}\,\{\mathbb{F}_n(t) - t\}\, w(t)
\end{aligned}
$$

Identify $\mathcal{F} = \{f_t \mid t \in [0, 1]\}$ with $[0, 1]$.

For a Brownian bridge B, do we have weak convergence in $\ell^\infty([0, 1])$

$$
\left(\sqrt{n}\,\{\mathbb{F}_n(t) - t\}\, w(t)\right)_{t\in[0,1]} \overset{?}{\rightsquigarrow} (B(t)w(t))_{t\in[0,1]}\,, \qquad n \to \infty
$$

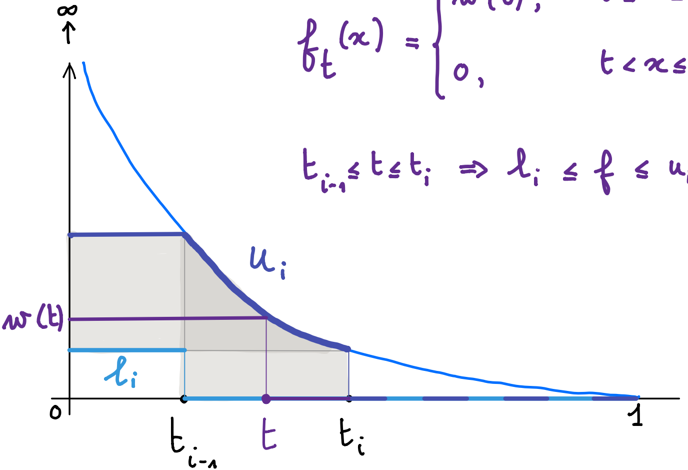
If yes, then construct tail-sensitive Kolmogorov–Smirnov test statistics

$$
\sup_{t\in[0,1]} \left| \sqrt{n}\,\{\mathbb{F}_n(t) - t\}\, w(t) \right|
$$

via w such that $w(t) \to \infty$ as $t \to 0$ or $t \to 1$

$$f_t(x) = \begin{cases} w(t), & 0 \le x \le t \\ 0, & t < x \le 1 \end{cases}$$

$$t_{i-1} \le t \le t_i \Rightarrow \ell_i \le f \le u_i$$



Assume $w : [0,1] \to [0,\infty]$ is decreasing, $w(0) = \infty$, $w(1) = 0$, and $\int_0^1 w^2(t)\,dt < \infty$.

Fix $\varepsilon > 0$. Find a sufficiently fine grid

$$0 = t_0 < t_1 < \ldots < t_m = 1$$

such that the brackets $[l_i, u_i]$ defined by

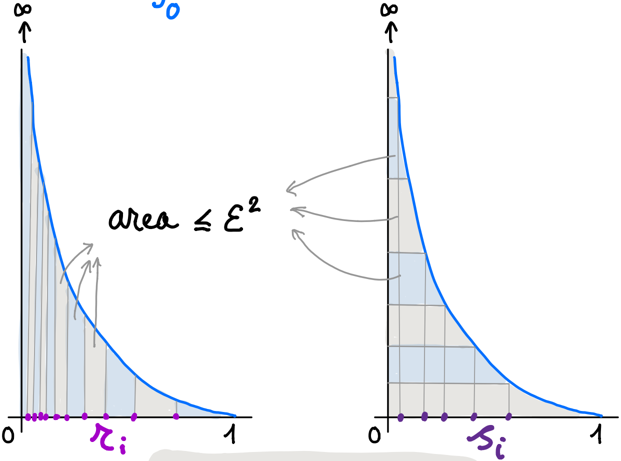
$$\begin{aligned} l_i(x) &= w(t_i)\mathbb{1}_{[0,t_{i-1}]}(x) \\ u_i(x) &= w(t_{i-1})\mathbb{1}_{[0,t_{i-1}]}(x) + w(x)\mathbb{1}_{(t_{i-1},t_i]}(x) \end{aligned}$$

have $L_2(P)$-size $[\int_{[0,1]}(u_i - l_i)^2]^{1/2} \leqslant \varepsilon$. The number m of points t_i needed is

$$N_{[]}(\varepsilon, \mathcal{F}, L_2(P)) = \mathrm{O}(1/\varepsilon^2)$$

As $\int_0^\delta \sqrt{\log(1/\varepsilon)}\,d\varepsilon < \infty$, the bracketing integral $J_{[]}(\delta, \mathcal{F}, L_2(P))$ is finite. □

$$\int_0^1 w^2(t)\,dt < \infty$$



$$\{t_i\} = \{r_i\} \cup \{s_i\}$$

Example: Parametric class

On general $\mathcal{X}$, consider $\mathcal{F} = \{f_\theta \mid \theta \in \Theta\}$ for bounded $\Theta \subset \mathbb{R}^d$.

Suppose there exists $m \in L_2(P)$ such that

$$\forall x \in \mathcal{X}, \forall \theta_1, \theta_2 \in \Theta, \qquad \left| f_{\theta_1}(x) - f_{\theta_2}(x) \right| \leqslant m(x) \, \|\theta_1 - \theta_2\|$$

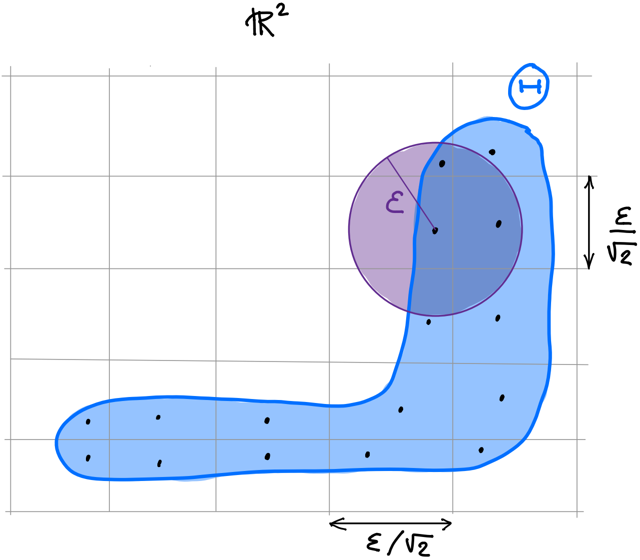
Claim: There exists $K > 0$ depending only on Θ, d, and m such that

$$\forall 0 < \varepsilon < \operatorname{diam} \Theta, \qquad N_{[]}\left(\varepsilon, \mathcal{F}, L_2(P)\right) \leqslant K \left(\frac{\operatorname{diam} \Theta}{\varepsilon} \right)^d$$

Proof: Find grid $\{\theta_i\}_{i=1}^N \subset \Theta$ such that ε-balls with centers θ_i cover Θ.

For $\varepsilon > 0$, cover $\mathcal{F}$ by brackets $[l_i, u_i] = [f_{\theta_i} - \varepsilon m, f_{\theta_i} + \varepsilon m]$.

Grid size N can be bounded by $O\left((\operatorname{diam} \Theta / \varepsilon)^d\right)$ as $\varepsilon \downarrow 0$. □



$\mathbb{R}^2$
Θ
ε
$\frac{\varepsilon}{\sqrt{2}}$
$\varepsilon/\sqrt{2}$

Mean absolute deviation

Let $X_1, X_2, \ldots$ be iid P over $\mathcal{X} = \mathbb{R}$; assume $\mathrm{E}[X^2] < \infty$.
Mean absolute deviation:

$$M_n = \frac{1}{n} \sum_{i=1}^{n} \left| X_i - \overline{X}_n \right|$$

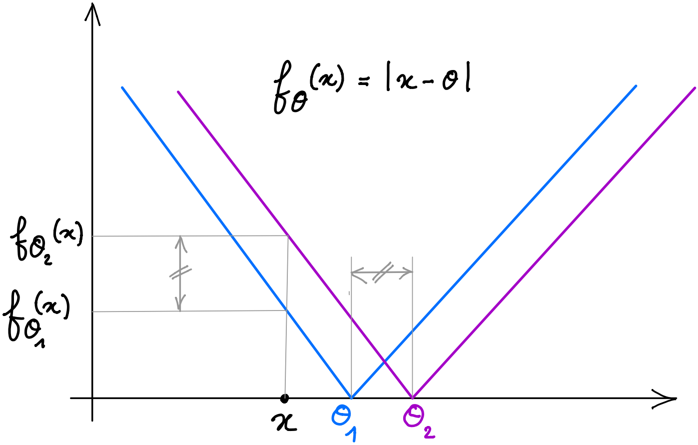
Asymptotic distribution of $\sqrt{n}(M_n - \mathrm{E}[|X - \mu|])$? Define

$$\forall \theta, x \in \mathbb{R}, \qquad f_\theta(x) = |x - \theta|$$

Writing $\mu = \mathrm{E}[X]$, we have

$$\begin{aligned} \sqrt{n}\,(M_n - \mathrm{E}[|X - \mu|]) &= \sqrt{n}\left(\mathbb{P}_n f_{\overline{X}_n} - P f_\mu\right) \\ &= \mathbb{G}_n f_{\overline{X}_n} + \sqrt{n}\left(P f_{\overline{X}_n} - P f_\mu\right) \end{aligned}$$

How to handle the two terms?


$f_\theta(x) = |x - \theta|$
$f_{\theta_2}(x)$
$f_{\theta_1}(x)$
x
θ_1
θ_2

1. Put $\mathcal{F} = \{f_\theta \mid \theta \in \Theta\}$ with $\Theta = (\mu - 1, \mu + 1)$. Then
   - $\overline{X}_n \xrightarrow{\mathrm{P}} \mu \in \Theta$ with large probability
   - $\mathcal{F}$ is P-Donsker by preceding example with $m(x) \equiv 1$
   - the map $\Theta \to L_2(P) : \theta \mapsto f_\theta$ is continuous

   $\mathbb{G}_n$ is asymptotically uniformly $L_2(P)$-equicontinuous in probability (lecture 2, slide 20):

$$\begin{aligned} \mathbb{G}_n f_{\overline{X}_n} &= \mathbb{G}_n f_\mu + o_{\mathrm{P}}(1) \\ &= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n |X_i - \mu| - \mathrm{E}[|X - \mu|]\right) + o_{\mathrm{P}}(1), \qquad n \to \infty \end{aligned}$$

2. If the cdf F of X is continuous at μ, the map $\theta \mapsto Pf_\theta = \mathrm{E}[|X - \theta|]$ is differentiable at $\theta = \mu$ with derivative $2F(\mu) - 1$.

   Delta method:

$$\sqrt{n}\left(Pf_{\overline{X}_n} - Pf_\mu\right) = \sqrt{n}\,(2F(\mu) - 1)\left(\overline{X}_n - \mu\right) + o_{\mathrm{P}}(1)$$

Summing up the two terms:

$$\sqrt{n}\,(M_n - \mathrm{E}[|X-\mu|]) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} h(X_i) - \mathrm{E}[h(X)]\right) + o_{\mathrm{P}}(1)$$
$$\xrightarrow{d} \mathcal{N}\left(0, \mathrm{var}(h(X))\right)$$

where

$$h(x) = |x-\mu| + (2F(\mu) - 1)\,x$$

□

If $F(\mu) = 1/2$, i.e., $\mathrm{E}[X] = \mu$ is also the median, the second term vanishes and it is as if we used μ rather than $\overline{X}_n$ in the definition of M_n.

Glivenko–Cantelli and Donsker Theorems

Bracketing entropy

Uniform entropy

VC-classes

Extension: Changing function classes

Covering numbers

- family $\mathcal{F}$ of measurable functions $f : \mathcal{X} \to \mathbb{R}$
- norm $\|\cdot\|$ on such functions

  usually $L_r(Q)$ for some $r \geqslant 1$ and some probability measure Q on $\mathcal{X}$

Definition: Covering number.

- *Covering number* of $\mathcal{F}$ with respect to $\|\cdot\|$:

$$N(\varepsilon, \mathcal{F}, \|\cdot\|) = \text{minimum number of } \varepsilon\text{-balls needed to cover } \mathcal{F},$$
$$\text{i.e., such that } \mathcal{F} = \bigcup_{i=1}^{N} \{g \in \mathcal{F} : \|g - f_i\| < \varepsilon\}$$

  Centers f_i need not belong to $\mathcal{F}$ but should have $\|f_i\| < \infty$.

- *Entropy (without bracketing)*:

$$\log N(\varepsilon, \mathcal{F}, \|\cdot\|)$$

Covering vs bracketing: if norm is such that $|f| \leqslant |g| \implies \|f\| \leqslant \|g\|$, then

$$N(\varepsilon, \mathcal{F}, \|\cdot\|) \leqslant N_{[]}(2\varepsilon, \mathcal{F}, \|\cdot\|)$$

Glivenko–Cantelli with covering

Glivenko–Cantelli theorem with random covering numbers

Suppose $\mathcal{F} \subset L_1(P)$ satisfies:

- $\mathcal{F}$ is P-measurable (see below)
- $\mathcal{F}$ has a P-integrable envelope $F : \mathcal{X} \to \mathbb{R}$

  $|f| \leqslant F$ for all $f \in \mathcal{F}$ and $PF < \infty$
- *random entropy condition*:

$$\frac{1}{n} \log N(\varepsilon \, \mathbb{P}_n F, \mathcal{F}, L_1(\mathbb{P}_n)) \xrightarrow{\mathrm{P}} 0, \qquad n \to \infty$$

then $\mathcal{F}$ is P-Glivenko–Cantelli, i.e.,

$$\sup_{f \in \mathcal{F}} |\mathbb{P}_n f - Pf| \xrightarrow{\mathrm{as}*} 0, \qquad n \to \infty$$

Sufficient that $\mathcal{F}$ is *pointwise measurable*, i.e., there exists countable $\mathcal{G} \subset \mathcal{F}$ such that for every $f \in \mathcal{F}$ there exists $g_m \in \mathcal{G}$ such that $\lim_{m\to\infty} g_m(x) = f(x)$ for all $x \in \mathcal{X}$ (van der Vaart and Wellner, 1996, p. 110).

Uniform covering numbers

- Recall covering number $N(\varepsilon, \mathcal{F}, \|\cdot\|)$
- Let $F : \mathcal{X} \to [0, \infty)$ be a cover of $\mathcal{F}$

Definition: Uniform covering.
For $r \geqslant 1$ and $\varepsilon > 0$, the *uniform covering number* is

$$\sup_Q N(\varepsilon \|F\|_{Q,r}, \mathcal{F}, L_r(Q))$$

- $\|F\|_{Q,r} = \|F\|_{L_r(Q)} = (QF^r)^{1/r} = (\int F^r \, dQ)^{1/r}$
- Supremum over all finitely discrete probability measures Q on $\mathcal{X}$ with $\|F\|_{Q,r} > 0$

The discrete measure $\mathbb{P}_n$ is among the Q in the supremum
$\Longrightarrow$ Random covering numbers are bounded by the uniform ones

Donsker with entropy

Donsker theorem with uniform entropy

Suppose $\mathcal{F} \subset L_2(P)$ satisfies

- $\mathcal{F}$ is "suitably measurable"
- $\mathcal{F}$ has a cover $F \in L_2(P)$
- $\mathcal{F}$ satisfies the *uniform entropy condition*:
  for some (and then for all) $\delta > 0$,

$$J(\delta, \mathcal{F}, L_2) := \int_0^\delta \sqrt{\log \sup_Q N(\varepsilon \|F\|_{Q,2}, \mathcal{F}, L_2(Q))} d\varepsilon < \infty$$

then $\mathcal{F}$ is P-Donsker, i.e., $\mathbb{G}_n \rightsquigarrow \mathbb{G}$ as $n \to \infty$ in $\ell^\infty(\mathcal{F})$.

For "suitably measurable": see Theorem 2.5.2 in van der Vaart and Wellner (1996).

Vapnik–Červonenkis index

Let $\mathcal{C}$ be a collection of subsets $C \subset \mathcal{X}$.

- Number of subsets A of $\{x_1, \ldots, x_n\} \subset \mathcal{X}$ *picked out* by $\mathcal{C}$:

$$\Delta_n(\mathcal{C}, x_1, \ldots, x_n) = \#\Big\{A \subset \{x_1, \ldots, x_n\} \,\Big|\, \exists C \in \mathcal{C} : A = C \cap \{x_1, \ldots, x_n\}\Big\}$$

- $\mathcal{C}$ *shatters* $\{x_1, \ldots, x_n\}$ if *every* $A \subset \{x_1, \ldots, x_n\}$ is picked out by $\mathcal{C}$:

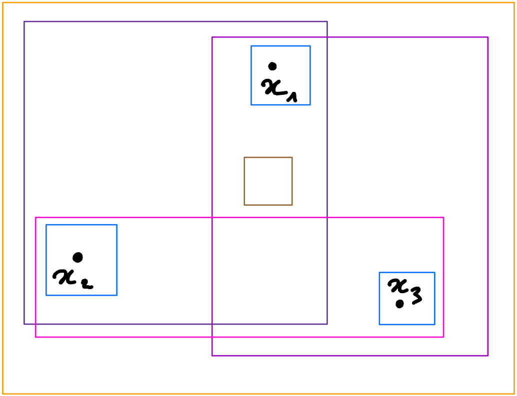
$$\Delta_n(\mathcal{C}, x_1, \ldots, x_n) = 2^n$$

Definition: VC-index. The *VC-index* of $\mathcal{C}$ is the smallest n such that *no set* of size n is shattered by $\mathcal{C}$:

$$V(\mathcal{C}) = \inf\Big\{n \in \mathbb{N} \,\Big|\, \forall x_1, \ldots, x_n : \Delta_n(\mathcal{C}, x_1, \ldots, x_n) < 2^n\Big\}$$

with $\inf \varnothing = \infty$

$\mathcal{C} = \{\text{rectangles in } \mathbb{R}^2, \text{ horizontal/vertical sides}\}$



All $2^3 = 8$ subsets of $\{x_1, x_2, x_3\}$ are picked out by $\mathcal{C}$

Example: rectangles in Euclidean space

In $\mathcal{X} = \mathbb{R}^d$, let $\mathcal{C}$ be the class of all rectangles

$$\prod_{j=1}^{d} ([a_j, b_j] \cap \mathbb{R})$$

with $-\infty \leqslant a_j \leqslant b_j \leqslant \infty$. Then

$$V(\mathcal{C}) \leqslant 2d + 1$$

(Could take open/closed rectangles too.)

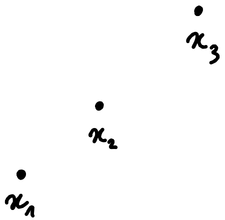
Proof.

For any $x_1, \ldots, x_n \in \mathbb{R}^d$ with $n = 2d + 1$, there exists $i \in \{1, \ldots, n\}$ such that x_i is "boxed in" by the other $n - 1$ points $\{x_k : k \neq i\}$.

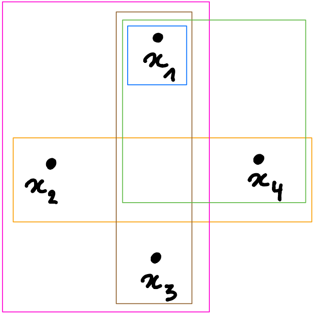
The set $A = \{x_i : k \neq i\}$ is *not* picked out by $\mathcal{C}$, i.e., there exists *no* rectangle $C \in \mathcal{C}$ such that

$$\{x_k : k \neq i\} = C \cap \{x_1, \ldots, x_n\}$$

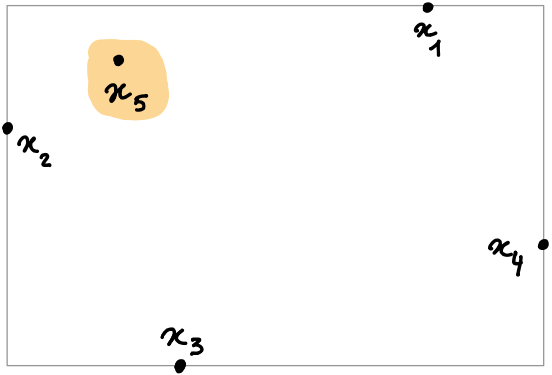
Hence, $\{x_1, \ldots, x_n\}$ is *not* shattered by $\mathcal{C}$. □



The set $A = \{x_1, x_3\}$ cannot be picked out.



All $2^4 = 16$ subsets of $\{x_1, \dots, x_4\}$ can be picked out.



With $n = 5$ points, there is always a point x_i such that $A = \{x_1, \dots, x_5\} \setminus \{x_i\}$ cannot be picked out $\Rightarrow$ $V(\mathcal{C}) = 5$

VC-classes of sets and functions

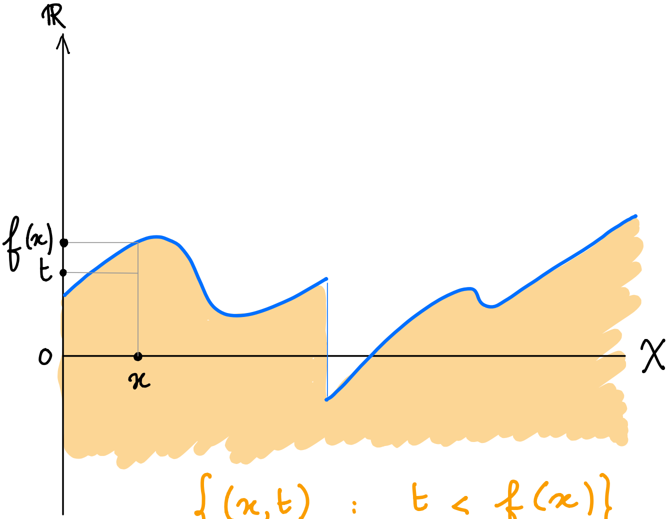
- Let $\mathcal{C}$ be a collection of subsets $C \subset \mathcal{X}$
- Let $\mathcal{F}$ be a collection of functions $f : \mathcal{X} \to \mathbb{R}$

Definition: VC-class.

- $\mathcal{C}$ is a *VC-class* if $V(\mathcal{C}) < \infty$
  There exists $n \in \mathbb{N}$ such that no set of size n is shattered by $\mathcal{C}$
- $\mathcal{F}$ is a *VC-class* if the collection of *subgraphs*

$$\Big\{\{(x,t) \in \mathcal{X} \times \mathbb{R} \mid t < f(x)\} \,\Big|\, f \in \mathcal{F}\Big\}$$

  is a VC-class in $\mathcal{X} \times \mathbb{R}$



$$\{(x,t) \; : \; t < f(x)\}$$

VC-classes and uniform covering numbers

Theorem: uniform covering numbers of VC-classes
There exists a universal constant $K > 0$ such that for any VC-class $\mathcal{F}$ of functions with envelope F, any $r \geqslant 1$, any $0 < \varepsilon < 1$,

$$\sup_Q N(\varepsilon \|F\|_{Q,r}, \mathcal{F}, L_r(Q)) \leqslant K\, V(\mathcal{F})\, (16e)^{V(\mathcal{F})} \left(\frac{1}{\varepsilon}\right)^{r(V(\mathcal{F})-1)}$$

the supremum being over all finitely discrete probability measures.

Proof via *Sauer's lemma*: if $\mathcal{C}$ is a VC-class of sets, then

$$\max_{x_1,\ldots,x_n \in \mathcal{X}} \Delta_n(\mathcal{C}, x_1, \ldots, x_n) \leqslant \sum_{j=0}^{V(\mathcal{C})-1} \binom{n}{j} = \mathrm{O}\left(n^{V(\mathcal{C})-1}\right), \qquad n \to \infty$$

Empirical processes indexed by VC-classes

VC-classes $\mathcal{F}$ are "small":

- the *random entropy condition* (slide 30) is trivially satisfied
- the uniform entropy integral (slide 32) converges:

$$\int_0^\delta \sqrt{\log(1/\varepsilon)}\, d\varepsilon < \infty$$

$\implies$ A VC-class $\mathcal{F}$ is Glivenko–Cantelli and Donsker provided it is suitably measurable and possesses an appropriate envelope F

Extensions to:

VC-hull classes $\mathcal{F}$: There exists a VC-class $\mathcal{G}$ such that every $f \in \mathcal{F}$ is the pointwise limit of a sequence $f_m = \sum_{i=1}^m \alpha_{mi} g_{mi}$ with $\sum_{i=1}^m |\alpha_{mi}| \leqslant 1$ and $g_{mi} \in \mathcal{G}$

VC-major classes $\mathcal{F}$: The collection $\mathcal{C} = \{C_{f,t} \mid f \in \mathcal{F}, t \in \mathbb{R}\}$ with $C_{f,t} = \{x \in \mathcal{X} \mid f(x) > t\}$ is VC in $\mathcal{X}$

(van der Vaart and Wellner, 1996, Sections 2.6.3 and 2.6.4)

Example: finite-dimensional vector spaces

Suppose there exists $f_1, \ldots, f_k : \mathcal{X} \to \mathbb{R}$ such that

$$\mathcal{F} \subset \{\lambda_1 f_1 + \cdots + \lambda_k f_k \mid \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}$$

Then $V(\mathcal{F}) \leqslant k + 2$.

Proof.

Let $n = k + 2$. Take any $(x_1, t_1), \ldots, (x_n, t_n) \in \mathcal{X} \times \mathbb{R}$. The vectors

$$v_f = (f(x_1) - t_1, \ldots, f(x_n) - t_n), \qquad f \in \mathcal{F}$$

lie in a $(k + 1)$-dimensional subspace $\mathbb{V}$ in $\mathbb{R}^{k+2}$. There exists $0 \neq a \in \mathbb{R}^{k+2}$ with at least one positive coordinate such that $a \perp \mathbb{V}$, i.e.,

$$\forall f \in \mathcal{F}, \qquad \sum_{i:a_i>0} a_i \,(f(x_i) - t_i) = \sum_{i:a_i<0} (-a_i)\,(f(x_i) - t_i)$$

The set $A = \{(x_i, t_i) \mid \text{those } i = 1, \ldots, n \text{ such that } a_i > 0\}$ is *not* picked out by a subgraph $\{(x, t) : t < f(x)\}$. □

Example: sets defined by polynomial equations

- If $\mathcal{F}$ is a VC-class of functions, then

$$\mathcal{C} = \Big\{ \{x \in \mathcal{X} \mid f(x) > 0\} \;\Big|\; f \in \mathcal{F} \Big\}$$

  is a VC-class of sets.

  Proof: Suppose $\mathcal{C}$ is not VC. For any n, there exists $\{x_1, \ldots, x_n\} \subset \mathcal{X}$ shattered by $\mathcal{C}$. Then $\{(x_1, 0), \ldots, (x_n, 0)\} \subset \mathcal{X} \times \mathbb{R}$ is shattered by the subgraphs $\{(x,t) : t < f(x)\}$ for $f \in \mathcal{F}$. Hence $\mathcal{F}$ is not a VC-class of functions, a contradiction. □

- The set $\mathcal{F}$ of polynomials on $\mathbb{R}^d$ up to a fixed degree m is a VC-class of functions by slide 42.
- Hence, the collection of half-spaces

$$\mathcal{C} = \Big\{ \Big\{ x \in \mathbb{R}^d \mid a_1 x_1 + \cdots + a_d x_d \leqslant b \Big\} \;\Big|\; a_1, \ldots, a_d, b \in \mathbb{R} \Big\}$$

  is a VC-class of sets. So are the ellipsoids, ...

Generating more examples through stability properties

If $\mathcal{C}$ and $\mathcal{D}$ are VC-classes of sets, then so are

- $\{C \cap D \mid C \in \mathcal{C}, D \in \mathcal{D}\}$
- $\{C \cup D \mid C \in \mathcal{C}, D \in \mathcal{D}\}$

Proof.
By Sauer's lemma, the number of subsets of any $\{x_1, \ldots, x_n\}$ picked out by intersections $C \cap D$ is only $O(n^{V(\mathcal{C})+V(\mathcal{D})-2}) < 2^n$. □

If $\mathcal{F}$ and $\mathcal{G}$ are VC-classes of functions, then so are

- $\{f \wedge g \mid f \in \mathcal{F}, g \in \mathcal{G}\}$
- $\{f \vee g \mid f \in \mathcal{F}, g \in \mathcal{G}\}$

Proof.
The subgraph of $f \wedge g$ is the intersection of the subgraphs of f and g. □

Counterexample: convex sets

In $\mathcal{X} = [-1, 1]^2$, the following class is *not* VC:

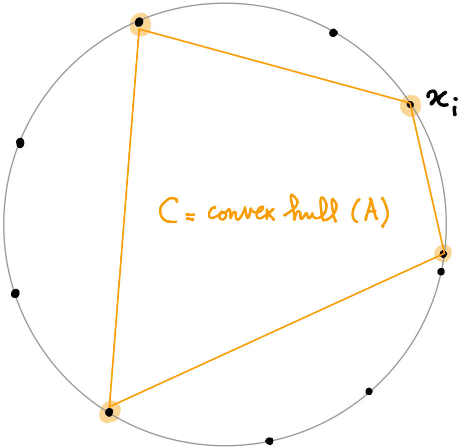
$$\mathcal{C} = \left\{\text{all convex polygons in } [-1, 1]^2\right\}$$

Proof.

Any set of n distinct points on the unit circle is shattered by $\mathcal{C}$. □

The class $\mathcal{F} = \{\mathbb{1}_C \mid C \in \mathcal{C}\}$ is *not* P-Glivenko–Cantelli or Donsker for the uniform distribution P on the unit circle: for all n,

$$\sup_{f \in \mathcal{F}} |\mathbb{P}_n f - Pf| = \sup_{C \in \mathcal{C}} \left|\mathbb{P}_n(C) - P(C)\right| = 1$$



Any $A \subset \{x_1, \ldots, x_n\}$ can be picked out

Glivenko–Cantelli and Donsker Theorems

Bracketing entropy

Uniform entropy

VC-classes

Extension: Changing function classes

Local empirical processes

Let $X_1, X_2, \ldots$ be iid uniform on $[0, 1]$.

Tail empirical process: for bandwidths $h = h_n \to 0$ such that $nh \to \infty$,

$$0 \leqslant t \mapsto \sqrt{nh}\left(\frac{1}{nh}\sum_{i=1}^{n} \mathbb{1}\{X_i \leqslant ht\} - t\right)$$

Example of a *local empirical process*:

- Density estimation
- Extreme-value theory (usual notation: $nh = k$)

Lindeberg–Feller central limit theorem: since $h \to 0$, the finite-dimensional distributions converge to those of a Wiener process $(W(t))_{t \geqslant 0}$

Centered Gaussian process such that $\mathrm{E}[W(s)W(t)] = s \wedge t$ for $s, t \geqslant 0$

Weak convergence in $\ell^\infty([0, M])$ for fixed $M > 0$?

Sums of independent stochastic processes

Alternative representation of the usual empirical process:

$$\mathbb{G}_n f = \sum_{i=1}^{n} \frac{f(X_i) - Pf}{\sqrt{n}} = \sum_{i=1}^{n} \Big(Z_{n,i}(f) - \mathrm{E}[Z_{n,i}(f)] \Big), \qquad f \in \mathcal{F}$$

Representation also encompasses tail empirical process on slide 48:

$$Z_{n,i}(t) = \frac{1}{\sqrt{nh}} \mathbb{1}\{X_i \leqslant ht\}, \qquad t \in [0, M] = \mathcal{F}$$

Convergence in $\ell^\infty(\mathcal{F})$?

Asymptotic tightness

Theorem. For each n, let $Z_{n,1}, \ldots, Z_{n,n}$ be independent stochastic processes with finite second moments indexed by a set $\mathcal{F}$. Suppose

1. $\lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{E}^* \left[\|Z_{n,i}\|_{\mathcal{F}} \mathbb{1}\{\|Z_{n,i}\|_{\mathcal{F}} > \lambda\} \right] = 0$ for all $\lambda > 0$
   Notation: $\|Z_{n,i}\|_{\mathcal{F}} = \sup_{f\in\mathcal{F}} |Z_{n,i}(f)|$
2. There exists $c > 0$ and, for all sufficiently small $\varepsilon > 0$, a covering $\mathcal{F} = \bigcup_{j=1}^{N_\varepsilon} \mathcal{F}_{\varepsilon,j}$ such that, for every set $\mathcal{F}_{\varepsilon,j}$ and every n,

$$\sum_{i=1}^{n} \mathrm{E}^* \left[\sup_{f,g\in\mathcal{F}_{\varepsilon,j}} \left| Z_{n,i}(f) - Z_{n,i}(g) \right|^2 \right] \leqslant c\varepsilon^2$$

3. For some $\delta > 0$, we have $\int_0^\delta \sqrt{\log N_\varepsilon}\, d\varepsilon < \infty$

Then $\sum_{i=1}^{n} \{Z_{n,i} - \mathrm{E}(Z_{n,i})\}$ is asymptotically tight in $\ell^\infty(\mathcal{F})$.
It converges weakly provided finite-dimensional distributions do so.

Corollary to Theorem 2.11.9 in van der Vaart and Wellner (1996)

Example: bracketing

Empirical process $\mathbb{G}_n f = \sqrt{n}(\mathbb{P}_n f - Pf)$ indexed by $f \in \mathcal{F} \subset L_2(P)$ and

$$J_{[]}(\delta, \mathcal{F}, L_2(P)) < \infty$$

Then $\mathcal{F}$ is P-Donsker by slide 16.
Alternatively, check conditions on slide 50 for $Z_{ni}(f) = n^{-1/2} f(X_i)$:

1. Cover $\mathcal{F}$ by N_1 brackets $[l_j, u_j]$ with $P[(u_j - l_j)^2] \leqslant 1$, then

$$\sup_{f \in \mathcal{F}} |f(x)| \leqslant \max_{j=1,\dots,N(1)} \max(|l_j(x)|, |u_j(x)|) =: F(x)$$

$$\|Z_{n,i}\|_{\mathcal{F}} \leqslant n^{-1/2} F(X_i)$$

   Since $F \in L_2(P)$, the condition follows from Markov's inequality

2. Cover $\mathcal{F}$ by N_ε brackets $[l_j, u_j] = \mathcal{F}_{\varepsilon,j}$ with $P[(u_j - l_j)^2] \leqslant \varepsilon^2$. Then

$$\sup_{f,g \in [l_j, u_j]} \left| Z_{n,i}(f) - Z_{n,i}(g) \right|^2 \leqslant \frac{1}{n} \left| u_j(X_i) - l_j(X_i) \right|^2$$

3. Since $J_{[]}(\delta, \mathcal{F}, L_2(P)) < \infty$.

Example: tail empirical process

Check three conditions on slide 50 for

$$Z_{n,i}(t) = (nh)^{-1/2}\mathbb{1}\{X_i \leqslant ht\}, \qquad t \in [0, M] = \mathcal{F}$$

1. Trivial, since $0 \leqslant \|Z_{n,i}\|_{\mathcal{F}} \leqslant (nh)^{-1/2} \to 0$
2. Construct a grid $0 = t_0 < t_1 < \ldots < t_{N_\varepsilon} = M$ with mesh $\varepsilon > 0$ and put

$$\mathcal{F}_{\varepsilon,j} = [t_{j-1}, t_j], \qquad j = 1, \ldots, N_\varepsilon$$

3. $N_\varepsilon = \mathrm{O}(1/\varepsilon)$ as $\varepsilon \downarrow 0$, hence $\int_0^\delta \sqrt{\log N_\varepsilon}\, d\varepsilon < \infty$

□

Summary

Empirical process indexed by family $\mathcal{F} \subset L_1(P)$ or $L_2(P)$ of functions $f : \mathcal{X} \to \mathbb{R}$

- P-Glivenko–Cantelli: uniform large of law numbers

$$\sup_{f \in \mathcal{F}} |\mathbb{P}_n f - Pf| \overset{\text{as}*}{\to} 0$$

- P-Donsker: uniform central limit theorem

$$\mathbb{G}_n = \sqrt{n}(\mathbb{P}_n - P) \rightsquigarrow \mathbb{G} \qquad \text{in } \ell^\infty(\mathcal{F})$$

Seek asymptotic tightness $\implies$ techniques to control the complexity of $\mathcal{F}$:

- bracketing numbers
- random and uniform covering numbers
  - VC-classes of sets and functions

References

van der Vaart, A. W. (1998). *Asymptotic Statistics*. Cambridge: Cambridge University Press.

van der Vaart, A. W. and J. A. Wellner (1996). *Weak Convergence and Empirical Processes. With Applications to Statistics*. New York: Springer Sciences+Business Media.