Statistical models and methods for spatial point processes

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Lectures:

1. Intro to point processes and moment measures
2. The Poisson process
3. Cox and cluster processes
4. Estimating functions
5. The conditional intensity and Markov point processes

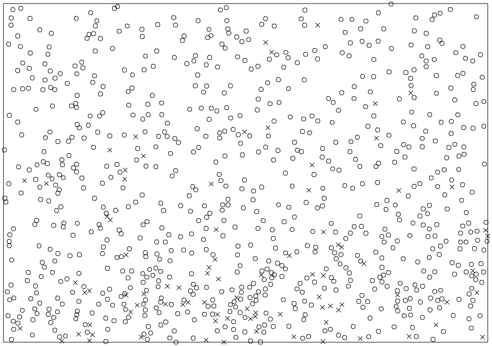
5. References

Aim: overview of

- spatial point process theory
- statistics for spatial point processes with emphasis on estimating equation inference
- not comprehensive: the most fundamental topics and my favorite things.
- all methods in Section 1-5 implemented in R package `spatstat`

Mucous membrane cells

Centres of cells in mucous membrane:



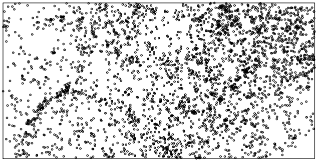
Repulsion due to physical extent of cells

Inhomogeneity - lower intensity in upper part

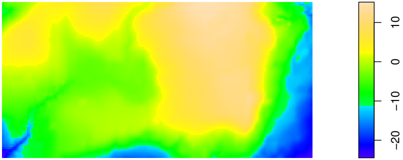
Bivariate - two types of cells

Same type of inhomogeneity for two types ?

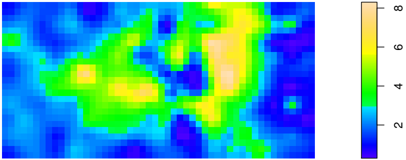
Data example: *Capparis Frondosa*



- *observation window W* $= 1000\text{ m} \times 500\text{ m}$
- seed dispersal$\Rightarrow$ *clustering*
- environment $\Rightarrow$ *inhomogeneity*



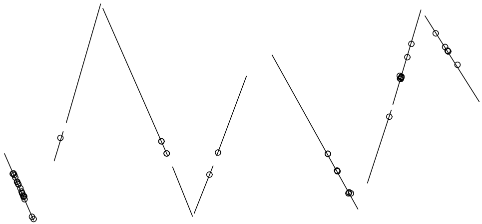
Elevation



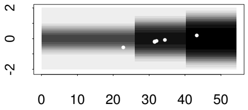
Potassium content in soil.

Objective: quantify dependence on environmental variables and clustering

Whale positions



Close up:



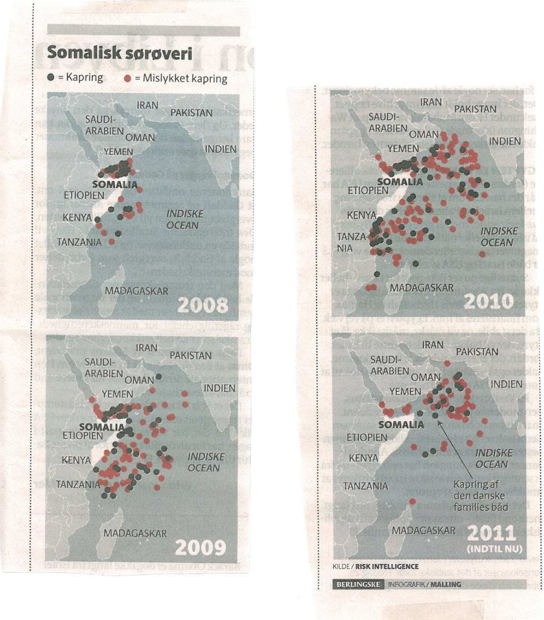
Aim: estimate whale intensity λ

Observation window W = narrow strips around transect lines

Varying detection probability: inhomogeneity (thinning)

Variation in prey intensity: clustering

Somalian pirates - two-type space-time

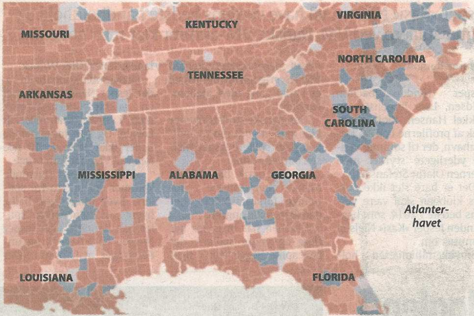


Cotton plantations in the Deep South

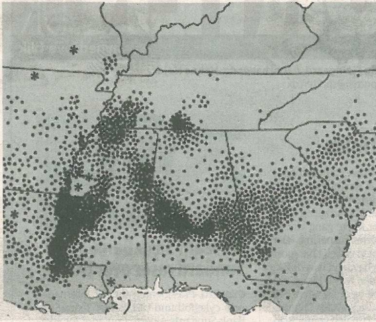
AMERIKANSKE TILSTANDE

Blog med Berlingske Tidendes USA-korrespondent Poul Høi.
www.usablog.blogs.berlingske.dk

Præsidentvalget i 2008 Republikanske amter Demokratiske amter



Placeringen af bomuldsplantagerne i 1860



Slaveri og politik i USA i dag

Den amerikanske borgerkrig influerer stadig på amerikansk politik. De tidligere slavestater i Syden er mere konservative end det øvrige USA, og det gjorde sig også gældende ved præsidentvalget i 2008. Her vandt Barack Obama noget nær en jordskredssejr, mens de fleste af de gamle slavestater stemte på hans modstander.
Barack Obama vandt dog enkelte amter i sydstaterne, nemlig de amter, hvor de fleste slaveplantager i sin tid lå, og dér hvor de fleste sorte vælgere i dag bor.

Forskere offentliggjorde ovenstående to kort. De
bomuldsplantagerne befandt sig i 1860, og det a
sydstaterne, som Barack Obama vandt i 2008, og
dende. På 2008-kortet er de republikanske amter
demokratiske med blåt.

What is a spatial point process ?

Definitions:

1. a locally finite random subset $\mathbf{X}$ of $\mathbb{R}^2$ ($\#(\mathbf{X} \cap A)$ finite for all bounded subsets $A \subset \mathbb{R}^2$)
2. stochastic process of count variables $\{N(B)\}_{B \in \mathcal{B}_0}$ indexed by bounded Borel sets $\mathcal{B}_0$.
3. a random counting measure N on $\mathbb{R}^2$

Equivalent provided no multiple points: ($N(A) = \#(\mathbf{X} \cap A)$)

This course: appeal to 1. and skip measure-theoretic details.

In practice distribution specified by an explicit construction (second and third lecture) or in terms of a probability density (second and fifth lecture).

Moments of a spatial point process

Fundamental characteristics of point process: mean and covariance of counts $N(A) = \#(\mathbf{X} \cap A)$.

Intensity measure μ:

$$\mu(A) = \mathbb{E}N(A), \quad A \subseteq \mathbb{R}^2$$

In practice often given in terms of *intensity function*

$$\mu(A) = \int_A \rho(u)\mathrm{d}u$$

Infinitesimal interpretation: $N(A)$ binary variable (presence or absence of point in A) when A very small. Hence

$$\rho(u)\mathrm{d}A \approx \mathbb{E}N(A) \approx P(\mathbf{X} \text{ has a point in A})$$

Second-order moments

Second order factorial moment measure:

$$\alpha^{(2)}(A \times B) = \mathrm{E} \sum_{u,v \in \mathbf{X}}^{\neq} \mathbf{1}[u \in A,\ v \in B] \qquad A, B \subseteq \mathbb{R}^2$$
$$= \int_A \int_B \rho^{(2)}(u,v)\,\mathrm{d}u\,\mathrm{d}v$$

where $\rho^{(2)}(u,v)$ is the *second order product density*

Infinitesimal interpretation of $\rho^{(2)}$:

$$\rho^{(2)}(u,v)dAdB \approx P(\mathbf{X} \text{ has a point in each of } A \text{ and } B)$$

$(u \in A\ , v \in B)$

Second moment vs. second factorial moment measure

Second moment measure

$$\mu^{(2)}(A \times B) = \mathbb{E}N(A)N(B) = \alpha^{(2)}(A \times B) + \mathbb{E}\sum_{u \in \mathbf{X}} 1[u \in A \cap B]$$

Hence due to "diagonal terms" in sum not absolutely continuous.

Campbell formulae

By definition of intensity function and product density and the standard proof we obtain the useful Campbell formulae:

$$\mathbb{E} \sum_{u \in \mathbf{X}} h(u) = \int h(u)\rho(u)\mathrm{d}u$$

$$\mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} h(u,v) = \iint h(u,v)\rho^{(2)}(u,v)\mathrm{d}u\mathrm{d}v$$

Pair correlation function

Pair correlation tendency to cluster/repel relative to case of independent points:

$$g(u,v) = \frac{\rho^{(2)}(u,v)}{\rho(u)\rho(v)} = \frac{P(\mathbf{X} \text{ has a point in each of } A \text{ and } B)}{P(\mathbf{X} \text{ has a point in A})P(\mathbf{X} \text{ has a point in B})}$$

$$= 1 \text{ if independence (Poisson process, next section)}$$

Let $\rho(u|v)$ denote intensity of $\mathbf{X}$ given $v \in \mathbf{X}$ ('Palm' intensity). Then

$$g(u,v) = \frac{\rho(u|v)}{\rho(u)}$$

Covariance and pair correlation function

$$\mathbb{C}\mathrm{ov}[N(A), N(B)] = \int_{A \cap B} \rho(u)\mathrm{d}u + \int_A \int_B \rho(u)\rho(v)(g(u,v) - 1)\mathrm{d}u\mathrm{d}v \tag{1}$$

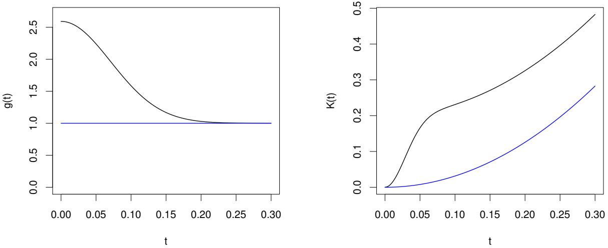
$$= \text{Poisson variance} + \text{additional/less variance due to interaction}$$

K-function

$$K(t) = \int_{\|h\| \le t} g(h) \mathrm{d}h$$

(provided $g(u, v) = g(u - v)$ i.e. **X** *second-order reweighted stationary*)

Examples of pair correlation and K-functions:



Estimation and interpretation of $K(t)$

Unbiased estimate of K-function (W observation window):

$$\hat{K}(t) = \sum_{u,v \in \mathbf{X} \cap W} \frac{1[0 < \|u - v\| \leq t]}{\rho(u)\rho(v)} e_{u,v}$$

($e_{u,v}$ edge correction factor)

In the homogeneous case (constant intensity ρ) $K(t)$ has interpretation as conditional expectation:

$$\rho K(t) = \mathbb{E}[\text{number of further points within distance } t \text{ of } u | u \in \mathbf{X}]$$

Exercises

1. Show that the covariance between counts $N(A)$ and $N(B)$ is

$$\mathbb{C}\mathrm{ov}[N(A), N(B] = \mu(A \cap B) + \alpha^{(2)}(A \times B) - \mu(A)\mu(B)$$

2. Verify covariance formula (1) (covariance in terms of pair correlation function).
3. Show that in the isotropic case ($g(u, v) = g(\|u - v\|)$), $K'(r) = 2\pi r g(r)$.
4. Show that

$$K(t) := \int_{\mathbb{R}^2} 1[\|u\| \leq t] g(u) \mathrm{d}u = \frac{1}{|B|} \mathbb{E} \sum_{\substack{u \in \mathbf{X} \cap B \\ v \in \mathbf{X}}}^{\neq} \frac{1[\|u - v\| \leq t]}{\rho(u)\rho(v)}$$

(Hint: use the Campbell formula)

5. Show that the following estimate is unbiased:

$$\hat{K}(t) = \sum_{u,v \in \mathbf{X} \cap W}^{\neq} \frac{1[\|u-v\| \leq t]}{\rho(u)\rho(v)|W \cap W_{u-v}|}$$

where W_{u-v} translated version of W (assume $|W \cap W_h| > 0$ for $\|h\| \leq t$).

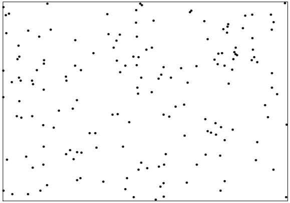
The Poisson process

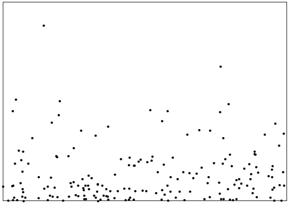
Assume μ locally finite measure on $\mathbb{R}^2$ with density ρ.

$\mathbf{X}$ is a Poisson process with intensity measure μ if for any bounded region B with $\mu(B) > 0$:

1. $N(B) \sim \text{Poisson}(\mu(B))$
2. Given $N(B)$, points in $\mathbf{X} \cap B$ i.i.d. with density $\propto \rho(u)$, $u \in B$

$B = [0,1] \times [0,0.7]$:



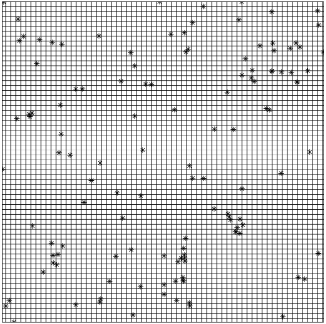


Homogeneous: $\rho = 150/0.7$ Inhomogeneous: $\rho(x, y) \propto \mathrm{e}^{-10.6y}$

Existence of Poisson process on $\mathbb{R}^2$: use definition on disjoint partitioning $\mathbb{R}^2 = \cup_{i=1}^{\infty} B_i$ of bounded sets B_i.

Homogeneous Poisson process as limit of Bernouilli trials

Consider disjoint subdivision $W = \cup_{i=1}^{n} C_i$ where $|C_i| = |W|/n$. With probability $\rho|C_i|$ a uniform point is placed in C_i.



Number of points in subset A is $b(n|A|/|W|, \rho|W|/n)$ which converges to a Poisson distribution with mean $\rho|A|$.

Hence, Poisson process default model when points occur independently of each other.

Characterization in terms of void probabilities

The distribution of any point process $\mathbf{X}$ is uniquely determined by the void probabilities $P(\mathbf{X} \cap B = \emptyset)$, for bounded subsets $B \subseteq \mathbb{R}^2$.

Intuition: consider very fine subdivision of observation window – then at most one point in each cell and (joint) probabilities of absence/presence determined by void probabilities.

Hence, a point process $\mathbf{X}$ with intensity measure μ is a Poisson process if and only if

$$P(\mathbf{X} \cap B = \emptyset) = \exp(-\mu(B))$$

for any bounded subset B.

Distribution and moments of Poisson process

$\mathbf{X}$ a Poisson process on S with $\mu(S) = \int_S \rho(u)\mathrm{d}u < \infty$ and F set of finite point configurations in S.

Examples of F: all point configurations with total number of points in a given interval, point configurations where all pairs of points separated by distance δ,...

By definition of a Poisson process and law of total probability

$$P(\mathbf{X} \in F) \\ = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu(S)}}{n!} \int_{S^n} 1[\{x_1, x_2, \dots, x_n\} \in F] \prod_{i=1}^{n} \rho(x_i)\mathrm{d}x_1 \dots \mathrm{d}x_n \quad (2)$$

Similarly,

$$\mathbb{E}h(\mathbf{X}) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu(S)}}{n!} \int_{S^n} h(\{x_1, x_2, \dots, x_n\}) \prod_{i=1}^{n} \rho(x_i)\mathrm{d}x_1 \dots \mathrm{d}x_n$$

Independent scattering:

- $\rho^{(2)}(u, v) = \rho(u)\rho(v)$ and $g(u, v) = 1$ (exercise)
- $\mathbb{C}\mathrm{ov}[N(A), N(B)] = \int_{A \cap B} \rho(u)\mathrm{d}u$
- $A, B \subseteq \mathbb{R}^2$ disjoint $\Rightarrow \mathbf{X} \cap A$ and $\mathbf{X} \cap B$ independent

Proof of independent scattering (finite case)

Consider bounded and disjoint $A, B \subseteq \mathbb{R}^2$.

$\mathbf{X} \cap (A \cup B)$ Poisson process.
Hence

$$
\begin{aligned}
& P(\mathbf{X} \cap A \in F, \mathbf{X} \cap B \in G) \quad (\mathbf{x} = \{x_1, \ldots, x_n\}) \\
= & \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu(A \cup B)}}{n!} \int_{(A \cup B)^n} 1[\mathbf{x} \cap A \in F, \mathbf{x} \cap B \in G] \prod_{i=1}^{n} \rho(x_i) \mathrm{d}x_1 \ldots \mathrm{d}x_n \\
= & \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu(A \cup B)}}{n!} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \int_{A^m} 1[\{x_1, x_2, \ldots, x_m\} \in F] \\
& \int_{B^{n-m}} 1[\{x_{m+1}, \ldots, x_n\} \in G] \prod_{i=1}^{n} \rho(x_i) \mathrm{d}x_1 \ldots \mathrm{d}x_n \\
= & \text{ (interchange order of summation and sum over } m \text{ and } k = n - m) \\
& P(\mathbf{X} \cap A \in F) P(\mathbf{X} \cap B \in G)
\end{aligned}
$$

Superpositioning and thinning

If $\mathbf{X}_1, \mathbf{X}_2, \ldots$ are independent Poisson processes (ρ_i), then *superposition* $\mathbf{X} = \cup_{i=1}^{\infty} \mathbf{X}_i$ is a Poisson process with intensity function $\rho(u) = \sum_{i=1}^{\infty} \rho_i(u)$ (provided ρ integrable on bounded sets).

Conversely: *Independent π-thinning* of Poisson process $\mathbf{X}$: independent retain each point u in $\mathbf{X}$ with probability $\pi(u)$. Thinned process $\mathbf{X}_{\text{thin}}$ and $\mathbf{X} \setminus \mathbf{X}_{\text{thin}}$ are independent Poisson processes with intensity functions $\pi(u)\rho(u)$ and $(1 - \pi(u))\rho(u)$.

(Superpositioning and thinning results most easily verified using void probability characterization of Poisson process, see M & W, 2003)

For general point process $\mathbf{X}$: thinned process $\mathbf{X}_{\text{thin}}$ has product density $\pi(u)\pi(v)\rho^{(2)}(u, v)$ - hence g and K invariant under independent thinning.

Density (likelihood) of a finite Poisson process

$\mathbf{X}_1$ and $\mathbf{X}_2$ Poisson processes on S with intensity functions ρ_1 and ρ_2 where $\int_S \rho_2(u)\mathrm{d}u < \infty$ and $\rho_2(u) = 0 \Rightarrow \rho_1(u) = 0$. Define $0/0 := 0$.
Then

$$
\begin{aligned}
&P(\mathbf{X}_1 \in F) \\
=&\sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu_1(S)}}{n!} \int_{S^n} 1[\mathbf{x} \in F] \prod_{i=1}^{n} \rho_1(x_i)\mathrm{d}x_1 \ldots \mathrm{d}x_n \quad (\mathbf{x} = \{x_1, \ldots, x_n\}) \\
=&\sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu_2(S)}}{n!} \int_{S^n} 1[\mathbf{x} \in F]\mathrm{e}^{\mu_2(S)-\mu_1(S)} \prod_{i=1}^{n} \frac{\rho_1(x_i)}{\rho_2(x_i)} \prod_{i=1}^{n} \rho_2(x_i)\mathrm{d}x_1 \ldots \mathrm{d}x_n \\
=&\mathbb{E}\big(1[\mathbf{X}_2 \in F]f(\mathbf{X}_2)\big)
\end{aligned}
$$

where

$$
f(\mathbf{x}) = \mathrm{e}^{\mu_2(S)-\mu_1(S)} \prod_{i=1}^{n} \frac{\rho_1(x_i)}{\rho_2(x_i)}
$$

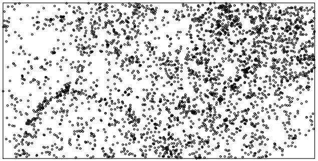
Hence f is a density of $\mathbf{X}_1$ with respect to distribution of $\mathbf{X}_2$.

In particular (if S bounded): $\mathbf{X}_1$ has density

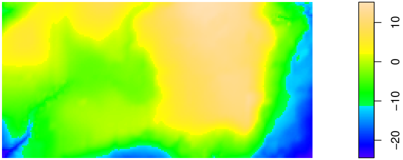
$$f(\mathbf{x}) = \mathrm{e}^{\int_S (1-\rho_1(u))\mathrm{d}u} \prod_{i=1}^{n} \rho_1(x_i)$$

with respect to unit rate Poisson process ($\rho_2 = 1$).

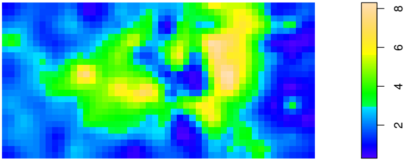
Back to the rain forest



- *observation window W* $= 1000\text{ m} \times 500\text{ m}$
- seed dispersal$\Rightarrow$ *clustering*
- environment $\Rightarrow$ *inhomogeneity*



Elevation



Potassium content in soil.

Objective: quantify dependence on environmental variables and clustering

Inhomogeneous Poisson process

Log linear intensity function

$$\rho(u;\beta) = \exp(z(u)\beta^{\mathsf{T}}), \quad z(u) = (1, z_{\text{elev}}(u), z_{\text{potassium}}(u), \ldots)$$

Estimate β from Poisson log likelihood (`spatstat`)

$$\sum_{u \in \mathbf{X} \cap W} z(u)\beta^{\mathsf{T}} - \int_W \exp(z(u)\beta^{\mathsf{T}})\mathrm{d}u \quad (W = \text{ observation window})$$

Model check using edge-corrected estimate of K-function

$$\hat{K}(t) = \sum_{u,v \in \mathbf{X} \cap W}^{\neq} \frac{1[\|u - v\| \leq t]}{\rho(u;\hat{\beta})\rho(v;\hat{\beta})|W \cap W_{u-v}|}$$

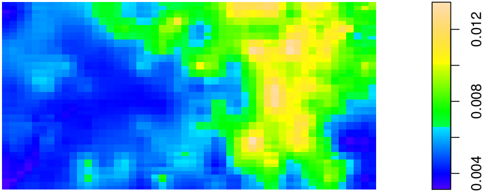
W_{u-v} translated version of W.

Capparis Frondosa and Poisson process ?

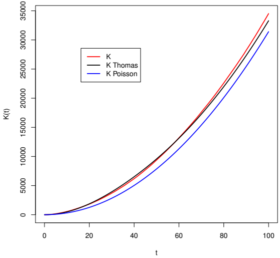
Fit model with covariates elevation, potassium,...

Fitted intensity function

$$\rho(u;\hat{\beta}) = \exp(\hat{\beta}z(u)^{\mathsf{T}})$$



Estimated K-function and $K(t) = \pi t^2$-function for Poisson process:



Not Poisson process - aggregation due to unobserved factors (e.g. seed dispersal)

Exercises

1. What is $K(t)$ for a Poisson process ?
2. Check that the Poisson expansion (2) indeed follows from the definition of a Poisson process.
3. How can you simulate an inhomogeneous Poisson process on a bounded region B in case $\rho(u)/\mu(B)$ is not a standard probability density ?
4. Show that $\rho^{(2)}(u, v) = \rho(u)\rho(v)$ for a Poisson process $\mathbf{X}$.

   (Hint: a) use that counts on disjoint subsets uncorrelated or b) compute second order factorial measure using the Poisson expansion for $\mathbf{X} \cap (A \cup B)$ for bounded $A, B \subseteq \mathbb{R}^2$.)

5. Assume that $\mathbf{X}$ has second order product density $\rho^{(2)}$ and show that g (and hence K) is invariant under independent thinning (note that a heuristic argument follows easily from the infinitesimal interpretation of $\rho^{(2)}$).

   (Hint: introduce random field $\mathbf{R} = \{R(u) : u \in \mathbb{R}^2\}$, of independent uniform random variables on $[0, 1]$, and independent of $\mathbf{X}$, and compute second order factorial measure for thinned process $\mathbf{X}_{\text{thin}} = \{u \in \mathbf{X} | R(u) \leq p(u)\}$.)

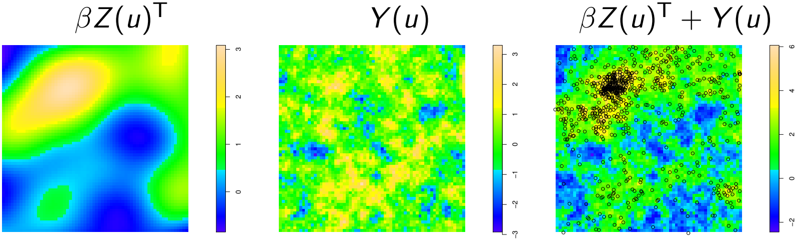
Cox processes

X is a *Cox process* driven by the random intensity function Λ if, conditional on $\Lambda = \lambda$, **X** is a Poisson process with intensity function λ.

Example: log Gaussian Cox process ("point process GLMM")

$$\log \Lambda(u) = \beta Z(u)^{\mathsf{T}} + Y(u)$$

where $\{Y(u)\}$ Gaussian random field.



Z: systematic variation Y: random clustering around peaks in Y

Wide range of covariance models available for Y: exponential, Gaussian, Matérn,...

Cox processes "bridge" between point processes and geostatistics.

Shot-noise Cox process

$$\Lambda(u) = \sum_{v \in \mathbf{C}} \gamma_v k(u - v)$$

where

- $\mathbf{C}$ homogeneous Poisson with intensity κ
- $k(\cdot)$ probability density.
- γ_v *iid* positive random variables independent of $\mathbf{C}$

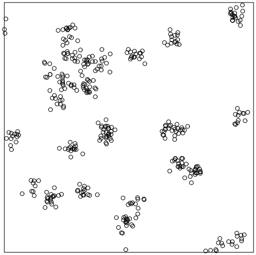
NB: equivalent to cluster process with parents $\mathbf{C}$, random cluster size γ_v and dispersal density k.

Inhomogeneous shot-noise:

$$\Lambda(u) = \exp[\beta Z(u)^{\mathsf{T}}] \sum_{v \in \mathbf{C}} \gamma_v k(u - v)$$

Inhomogeneous Thomas: inhomogeneous shot-noise with Gaussian $k(\cdot)$ and $\gamma_v = \alpha > 0$.

Cluster process: Inhomogeneous Thomas process



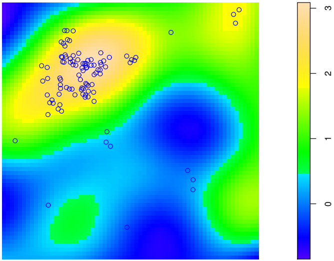
Parents stationary Poisson point process intensity κ

Poisson(α) number of offspring distributed around parents according to bivariate Gaussian density

Inhomogeneity: offspring survive according to probability

$$p(u) \propto \exp(Z(u)\beta^{\mathsf{T}})$$

depending on covariates (independent thinning).



Moments for Cox processes

Intensity function

$$\rho(u) = \mathbb{E}\Lambda(u)$$

Second-order product density

$$\rho^{(2)}(u,v) = \mathbb{E}\Lambda(u)\Lambda(v) = \mathbb{C}\mathrm{ov}[\Lambda(u),\Lambda(v)] + \rho(u)\rho(v)$$

$$\begin{aligned}
\mathbb{C}\mathrm{ov}[N(A),N(B)] &= \int_{A\cap B} \mathbb{E}\Lambda(u)\mathrm{d}u + \int_A\int_B \mathbb{C}\mathrm{ov}[\Lambda(u),\Lambda(v)]\mathrm{d}u\mathrm{d}v \\
&= \int_{A\cap B} \rho(u)\mathrm{d}u + \int_A\int_B \rho(u)\rho(v)[g(u,v)-1]\mathrm{d}u\mathrm{d}v \\
&= \text{ Poisson variance } + \text{ extra variance due to } \Lambda
\end{aligned}$$

(overdispersion relative to a Poisson process)

Common structure: log-linear model

Both log Gaussian and shot-noise Cox process of the form

$$\Lambda(u) = \Lambda_0(u) \exp[\beta Z(u)^\mathsf{T}]$$

where Λ_0 stationary non-negative reference process.

(interpretation: Cox process $\mathbf{X}$ independent inhomogeneous thinning of stationary $\mathbf{X}_0$ with random intensity function Λ_0).

Log-linear intensity (assume $\mathbb{E}\Lambda_0(u) = 1$)

$$\rho(u) = \mathbb{E}\Lambda(u) = \exp[\beta Z(u)^\mathsf{T}]$$

Pair correlation function ($\mathbb{E}\Lambda_0(u) = 1$):

$$g(h) = 1 + c_0(h) \quad c_0(h) = \mathbb{C}\mathrm{ov}[\Lambda_0(u), \Lambda_0(u+h)]$$

Specific models for $c_0(u-v) = \mathbb{C}\mathrm{ov}[\Lambda_0(u), \Lambda_0(v)]$

Log-Gaussian:

$$\Lambda_0(u) = \exp[Y(u)]$$

where Y Gaussian field.

Covariance (Laplace transform of normal distribution):

$$c_0(h) = \exp[\mathbb{C}\mathrm{ov}(Y(u), Y(u+h))] - 1$$

Shot-noise:

$$\Lambda_0(u) = \sum_{v \in C} \gamma_v k(u-v)$$

Covariance (convolution):

$$c_0(u-v) = \kappa\alpha^2 \int_{\mathbb{R}^2} k(u)k(u+h)\mathrm{d}u$$

$(\alpha = \mathbb{E}\gamma_v)$

normal-variance mixture Cox/cluster processes

Suppose kernel $k(\cdot)$ given by variance-gamma density.

Y variance-gamma if $Y = \sqrt{W}U$ where $W \sim \Gamma$ and $U \sim N_p(0, I)$
$\Rightarrow$ closed under convolution.

Then Matérn covariance function:

$$c_0(h) = \sigma_0^2 \frac{(\|h\|/\eta)^\nu K_\nu(\|h\|/\eta)}{2^{\nu-1}\Gamma(\nu)}$$

Suppose $k(\cdot)$ Cauchy density (W inverse-gamma)

$$k(u) = \frac{1}{2\pi\omega^2}[1 + (\|u\|/\omega)^2)]^{-3/2}$$

then

$$c_0(r) = \sigma_0^2[1 + (\|r\|/\eta)^2]^{-3/2}$$

Cauchy too ($\sigma_0^2 = \kappa\xi^2/(2\pi\eta)^2 \ \eta = 2\omega$)

Density of a Cox process

- Restricted to a bounded region W, the density is

$$f(\mathbf{x}) = \mathbb{E}\left[\exp\left(|W| - \int_W \Lambda(u)\,\mathrm{d}u\right) \prod_{u \in \mathbf{X}} \Lambda(u)\right]$$

- Not on closed form
- likelihood-based inference: MCMC or Laplace approximation (INLA for log Gaussian Cox processes)
- estimating equations based on closed form expressions for intensity and pair correlation

Exercises

1. For a Cox process with random intensity function Λ, show that

   $$\mathbb{V}\mathrm{ar} N(A) \geq \mathbb{E} N(A), \quad \rho(u) = \mathbb{E}\Lambda(u), \quad \rho^{(2)}(u,v) = \mathbb{E}[\Lambda(u)\Lambda(v)]$$

   (hint: use conditioning on Λ)
2. Show that a cluster process with Poisson(α) number of iid offspring is a Cox process with random intensity function

   $$\Lambda(u) = \alpha \sum_{v \in \mathbf{C}} k(u-v)$$

   (using notation from previous slide on cluster processes. Hint: use void probability characterisation and superposition result for Poisson process. Note: **C** can be any point process)
3. Compute the intensity and second-order product density for an inhomogeneous Thomas process. (Hint: interpret the Thomas process as a Cox process and use the Campbell formula)
4. Show that pair correlation for LCGP is $g(u,v) = \exp[\mathbb{C}\mathrm{ov}(Y(u), Y(v))]$

Maximum likelihood estimation for Poisson

Log likelihood for Poisson process with intensity function ρ_θ:

$$l(\theta) = \sum_{u \in \mathbf{X}} \log \rho_\theta(u) - \int_W \rho_\theta(u) \mathrm{d}u$$

Score (first derivative):

$$s(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta} l(\theta) = \sum_{u \in \mathbf{X}} \frac{\rho'_\theta(u)}{\rho_\theta(u)} - \int_W \rho'_\theta(u) \mathrm{d}u$$

Find $\hat{\theta}$ by solving $s(\theta) = 0$. Unique solution if observed information

$$-\frac{\mathrm{d}^2}{\mathrm{d}\theta^\mathsf{T} \mathrm{d}\theta} l(\theta)$$

positive definite.

Information matrix:

$$i(\theta) = -\mathbb{E}\frac{\mathrm{d}^2}{\mathrm{d}\theta^\mathsf{T}\mathrm{d}\theta} l(\theta)$$

Under weak regularity conditions,

$$\hat{\theta} \approx N(\theta, i(\theta)^{-1})$$

If Poisson process not appropriate due to clustering we might consider Cox/cluster processes but likelihood function is then hard to compute.

To move on, estimating function perspective is useful.

Estimating function

Estimating function: $e(\theta)$ $[e(\theta, \mathbf{X})]$ function of θ and data $\mathbf{X}$.

Parameter estimate $\hat{\theta}$ solution of

$$e(\theta) = 0$$

First order Taylor:

$$e(\theta) \approx S(\hat{\theta} - \theta)$$

where sensitivity:

$$S = -\mathbb{E}[\frac{\mathrm{d}}{\mathrm{d}\theta} e(\theta)]$$

minus expected derivative of $e(\theta)$

Using Taylor approximation: $\hat{\theta}$ approx. unbiased $\mathbb{E}\hat{\theta} = \theta$ if $e(\theta)$ unbiased $\mathbb{E}e(\theta) = 0$ (θ true value).

Moreover ('sandwich'-variance estimator):

$$\mathbb{V}\mathrm{ar}\hat{\theta} \approx S^{-1}\Sigma S^{-\mathsf{T}} \quad \Sigma = \mathbb{V}\mathrm{ar}e(\theta)$$

Note: in case of Poisson process and $e(\theta)$ equal to likelihood score, $S = \mathbb{V}\mathrm{ar}e(\theta) = i(\theta)$ whereby $\mathbb{V}\mathrm{ar}\hat{\theta} = i(\theta)^{-1}$.

How do we construct unbiased estimating functions involving $\mathbf{X}$ and θ ?

Composite likelihood

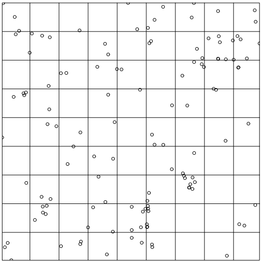
Disjoint subdivision $W = \cup_{i=1}^{m} C_i$ in 'cells' C_i.

$u_i \in C_i$ 'center' point.

Random indicator variables:

$$Y_i = 1[\mathbf{X} \text{ has a point in } C_i]$$

(presence/absence of points in C_i).



$$P(Y_i = 1) = |C_i| \rho_\theta(u_i)$$

Idea: form composite likelihoods based on Y_i, e.g.

$$\prod_i P(Y_i = 1)^{Y_i} (1 - P(Y_i = 1))^{1 - Y_i}$$

Consider limit when $|C_i| \to 0$.

Composite likelihood (in fact likelihood for Poisson):

$$\left[\prod_{u\in\mathbf{X}} \rho_\theta(u)\right] \exp\left[\int_W \rho_\theta(u)\mathrm{d}u\right]$$

Score:

$$e(\theta) = \sum_{u\in\mathbf{X}} \frac{\rho'_\theta(u)}{\rho_\theta(u)} - \int_W \rho'_\theta(u)\mathrm{d}u$$

unbiased estimating function by Campbell.

Sensitivity is equal to Information matrix for Poisson process.
Variance

$$\mathbb{V}\mathrm{ar}\, e(\theta) = \mathbb{V}\mathrm{ar} \sum_{u\in\mathbf{X}} \frac{\rho'_\theta(u)}{\rho_\theta(u)}$$

can be evaluated using second Campbell formula. Larger than $i(\theta)$ in case of Cox/cluster ($g_\theta(\cdot) > 1$).

Note: to evaluate sandwich estimator of variance

$$S^{-1}\mathbb{V}\mathrm{ar}e(\theta)S^{-\mathsf{T}}$$

of parameter estimates, we need estimate of pair correlation function (later).

Other issue:

- integral

  $$\int_W \rho'_\theta(u)\mathrm{d}u$$

  often not explicitly computable.

  Can be approximated fairly easy using numerical quadrature or Monte Carlo (later).

Estimation of pair correlation function

Suppose parametric model $g(\cdot;\psi)$ for pair correlation.

Some options:

1. minimum contrast estimation based on K-function.
2. second-order composite likelihood: composite likelihood based on indicators for joint occurrence of points in pairs of cells:

$$X_{ij} = 1[N_i > 0 \text{ and } N_j > 0]$$

Minimum contrast estimation for ψ

Computationally easy alternative if $\mathbf{X}$ second-order reweighted stationary so that K-function well-defined.

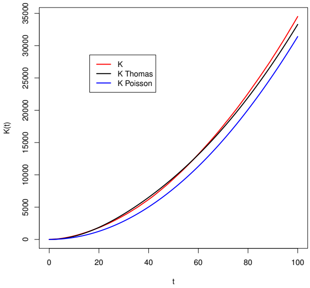
Estimate of K-function:

$$\hat{K}_\beta(t) = \sum_{u,v \in \mathbf{X} \cap W} \frac{1[0 < \|u-v\| \leq t]}{\rho(u;\beta)\rho(v;\beta)} e_{u,v}$$

Unbiased if β 'true' regression parameter.

Minimum contrast estimation: minimize squared distance between theoretical K and $\hat{K}$:

$$\hat{\psi} = \underset{\psi}{\operatorname{argmin}} \int_0^r \left(\hat{K}_{\hat{\beta}}(t) - K(t;\psi)\right)^2 \mathrm{d}t$$



Second-order composite likelihood

Consider indicators for *joint* occurrence of points in pairs of cells:

$$X_{ij} = 1[N_i > 0 \text{ and } N_j > 0]$$

with

$$\begin{aligned} P_{\beta,\psi}(X_{ij} = 1) &= \rho^{(2)}(u, v; \beta, \psi)|C_i||C_j| \\ &= \rho_\beta(u_i)\rho_\beta(v_j)g(u_i - u_j; \psi)|C_i||C_j| \end{aligned}$$

Second-order composite likelihood:

$$\mathrm{CL}_2(\beta, \psi) = \prod_{\substack{u,v \in \mathbf{X} \cap W \\ \|u-v\| \leq R}}^{\neq} \rho^{(2)}(u, v; \beta, \psi) \times \exp\left[- \iint_{\|u-v\| \leq R} \rho^{(2)}(u, v; \beta, \psi) \mathrm{d}u \mathrm{d}v \right]$$

NB: second-order reweighted stationarity (translation invariant pair correlation) not required.

In practice we plug in $\hat{\beta}$ from first order composite likelihood.

Two-step estimation

Obtain estimates $(\hat{\beta}, \hat{\psi})$ in two steps

1. obtain $\hat{\beta}$ using composite likelihood
2. obtain $\hat{\psi}$ using minimum contrast/second order composite likelihood (replacing β by $\hat{\beta}$ from first step)

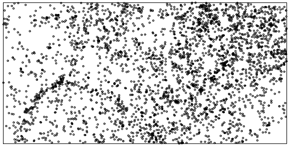
Implementation spatstat

Two-step estimation implemented in `spatstat` procedure `kppm`

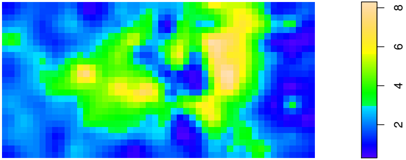
Options composite likelihood, quasi-likelihood, minimum contrast, second-order composite likelihood,...

Example: rain forest trees

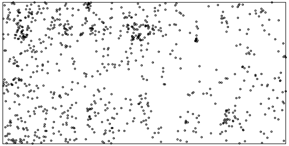
Capparis Frondosa



Potassium content in soil.



Loncocharpus Heptaphyllus



Covariates pH, elevation, gradient, potassium,...

Clustered point patterns: Cox point process natural model.

Objective: infer regression model $\rho_\beta(u) = \exp[\beta Z(u)^\mathsf{T}]$

Composite likelihood targeted at estimating intensity function.

Results with composite likelihood (and quasi-likelihood - later)

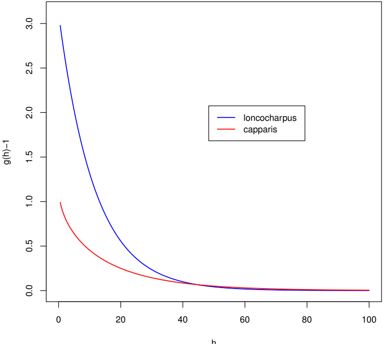
species		$\widehat{\beta}$
Loncocharpus	CL	$-6.49 - 0.021\texttt{Nmin} - 0.11\texttt{P} - 0.59\texttt{pH} - 0.11\texttt{twi}$ $(81.06^*, 7.45^*, 58.78, 282.89^*, 53.19^*) \times 10^{-3}$
	QL	$-6.49 - 0.023\texttt{Nmin} - 0.12\texttt{P} - 0.55\texttt{pH} - 0.084\texttt{twi}$ $(80.15^*, 6.95^*, 55.23^*, 266.10^*, 45.47) \times 10^{-3}$
Capparis	CL	$-5.07 + 0.028\texttt{ele} - 1.10\texttt{grad} + 0.0043\texttt{K}$ $(79.54^*, 9.98^*, 1200.36, 1.16^*) \times 10^{-3}$
	QL	$-5.10 + 0.019\texttt{ele} - 2.50\texttt{grad} + 0.0039\texttt{K}$ $(77.77^*, 8.86^*, 935.02^*, 1.02^*) \times 10^{-3}$

Estimated standard errors always smallest for QL. Covariate `grad` significant according to QL but not for CL.

Fitted pair correlation functions $g(\cdot)$ for Capparis and Loncocharpus

Use shot-noise Cox process with dispersal kernel given by variance-gamma density.

Then $g(h) - 1$ *Matérn covariance* function depending on smoothness/shape parameter ν.



Loncocharpus:
Matérn $\nu = 0.5$

Capparis:
Matérn $\nu = 0.25$

Optimality

Composite likelihood score

$$\sum_{u \in \mathbf{X}} \frac{\rho'_\beta(u)}{\rho_\beta(u)} - \int_W \rho'_\beta(u)\mathrm{d}u$$

optimal for Poisson (likelihood).

Which f makes

$$e_f(\beta) = \sum_{u \in \mathbf{X} \cap W} f(u) - \int_W f(u)\rho_\beta(u)\mathrm{d}u$$

optimal for Cox point process (positive dependence between points) ?

Optimal first-order estimating equation

Optimal choice of f: smallest variance

$$\mathbb{V}\mathrm{ar}\hat{\beta} = V_f = S_f^{-1} \Sigma_f S_f^{-\mathsf{T}}$$

where

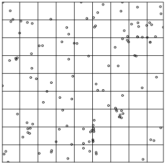
$$S_f = -\mathbb{E} \frac{\mathrm{d}}{\mathrm{d}\beta^{\mathsf{T}}} e_f(\beta) \quad \Sigma_f = \mathbb{V}\mathrm{ar} e_f(\beta)$$

Possible to obtain optimal f as solution of certain Fredholm integral equation.

Numerical solution of integral equation leads to estimating function of quasi-likelihood type.

Quasi-likelihood

Integral equation approximated using Riemann sum dividing W into cells C_i with representative points u_i.



Resulting estimating function is *quasi-likelihood*

$$(Y - \mu)V^{-1}D$$

based on

$$Y = (Y_1, \dots, Y_m), \quad Y_i = 1[\mathbf{X} \text{ has point in } C_i].$$

μ mean of Y:

$$\mu_i = \mathbb{E}Y_i = \rho_\beta(u_i)|C_i| \text{ and } D = \left[\mathrm{d}\mu(u_i)/\mathrm{d}\beta_l\right]_{il}$$

V covariance of Y (involves covariance of random intensity):

$$V_{ij} = \mathbb{C}\mathrm{ov}[Y_i, Y_j] = \mu_i 1[i = j] + \mu_i\mu_j[g(u_i, u_j) - 1]$$

Approximation of integral in composite likelihood

Issue: integral

$$\int_W \rho'(u)\mathrm{d}u$$

in composite likelihood typically not available in closed form.

Deterministic numerical quadrature:

1. resulting estimating function not unbiased
2. difficult to quantify resulting bias of parameter estimates.

Monte Carlo approximation of integral in composite likelihood

Let **D** 'quadrature/dummy' point process of intensity κ and independent of **X**.

By Campbell

$$\int_W \rho'(u)\mathrm{d}u = \mathbb{E} \sum_{u \in \mathbf{X} \cup \mathbf{D}} \frac{\rho'(u)}{\rho(u)+\kappa} \approx \sum_{u \in \mathbf{X} \cup \mathbf{D}} \frac{\rho'(u)}{\rho(u)+\kappa}$$

Idea: replace integrals in pseudo- or composite likelihood with unbiased estimates using **D**.

Advantages:

1. unbiased approximation $\Rightarrow$ still unbiased estimating function !
2. CLT available for approximation $\Rightarrow$ CLT for parameter estimates.

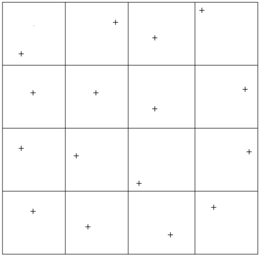
Dummy point process

Should be easy to simulate and mathematically tractable.

Possibilities:

1. Poisson process
2. binomial point process (fixed number of independent points)
3. stratified binomial point process

Stratified:



Approximate composite likelihood scores:

$$s(\theta) = \sum_{u \in \mathbf{X}} \frac{\rho_\theta^{'}(u)}{\rho_\theta(u)} - \sum_{u \in (\mathbf{X} \cup \mathbf{D})} \frac{\rho_\theta^{'}(u)}{\rho_\theta(u) + \kappa} \tag{3}$$

Note: of *logistic regression/case control* form with 'probabilities'

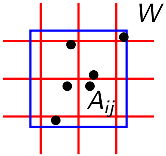
$$p(u) = \frac{\rho_\theta(u)}{\rho_\theta(u) + \kappa}$$

I.e. probabilities that $u \in \mathbf{X}$ given $u \in \mathbf{X} \cup \mathbf{D}$.

Hence computations straightforward with `glm()` software !

Monte Carlo and deterministic numerical quadrature implemented in spatstat procedure `ppm`

Asymptotic results - first order estimating function



Divide $\mathbb{R}^2$ into quadratic cells
$A_{ij} = [i, i+1[\times [j, j+1[$

Then

$$e_f(\beta) = \sum_{ij: A_{ij} \subseteq W} U_{ij}$$

where

$$U_{ij} = \sum_{u \in \mathbf{X} \cap A_{ij}} f_\beta(u) - \int_{A_{ij}} f_\beta(u) \rho_\beta(u) \mathrm{d}u$$

Assuming $\mathbf{X}$ is mixing, $\{U_{ij}\}_{ij}$ mixing random field and

$$|W|^{-1/2} e_f(\beta) \approx N(0, \Sigma_f)$$

(CLT for mixing random field $\{U_{ij}\}_{ij}$).

Asymptotic results cntd.

Estimate $\hat{\beta}$ solves

$$e_f(\beta) = 0$$

And (Taylor)

$$e_f(\beta) \approx |W| S_f(\hat{\beta} - \beta) \Leftrightarrow (\hat{\beta} - \beta) = |W|^{-1} S_f^{-1} e_f(\beta)$$

where

$$S_f = -\mathbb{E} \frac{\mathrm{d}}{\mathrm{d}\beta^{\mathsf{T}}} e_f(\beta)/|W|$$

It follows that

$$\hat{\beta} \approx N(\beta, V_f/|W|)$$

where

$$V_f = S_f^{-1} \Sigma_f S_f^{-\mathsf{T}}$$

Alternative: "infill" /increasing intensity-asymptotics

If $\mathbf{X}$ infinitely divisible (e.g. Poisson or Poisson-cluster) then

$$\mathbf{X} = \cup_{i=1}^{n} \mathbf{X}_i$$

where $\mathbf{X}_i$ *iid* and intensity of $\mathbf{X}$ is $\rho_\beta(u) = n\tilde{\rho}(u;\beta)$ where $\tilde{\rho}_\beta$ intensity of $\mathbf{X}_i$.

Thus

$$e_f(\beta) = \sum_{i=1}^{n} \Big[\sum_{u \in \mathbf{X}_i} f_\beta(u) - \int_W f_\beta(u)\tilde{\rho}(u;\beta)\mathrm{d}u \Big].$$

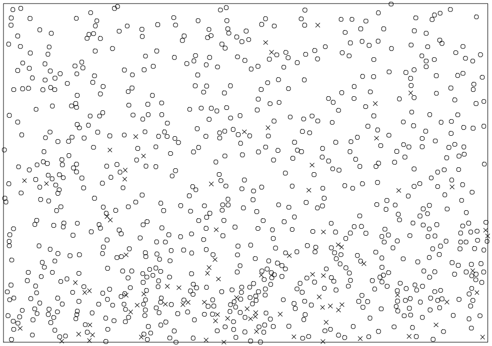
Ordinary CLT applies !

Exercises

1. Compute information matrix and variance of log likelihood score in case of a Poisson process with intensity function $\rho_\theta(\cdot)$.
2. Obtain expression for $\mathbb{V}\mathrm{ar}e(\theta)$ in terms of pair correlation function g in case of first order composite likelihood.
3. Check that the derivative of minimum contrast criterion and the score of the second order composite likelihood function are unbiased estimating functions when β is equal to the true value.
4. How can you partition a Poisson-cluster process $\mathbf{X}$ into a union $\cup_{i=1}^{n}\mathbf{X}_i$ of *iid* Poisson-cluster processes ?
5. show that the approximate composite likelihood score (3) is of logistic regression score form when the intensity is log linear.
6. Derive the second-order product density of a stratified binomial point process with one point in each cell.

Mucous membrane cells

Centres of cells in mucous membrane:



Repulsion due to physical extent of cells

Inhomogeneity - lower intensity in upper part

Bivariate - two types of cells

Same type of inhomogeneity for two types ?

Density with respect to a Poisson process

$\mathbf{X}$ on bounded S has density f with respect to unit rate Poisson $\mathbf{Y}$ if

$$P(\mathbf{X} \in F) = \mathbb{E}\big(1[\mathbf{Y} \in F] f(\mathbf{Y})\big)$$
$$= \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-|S|}}{n!} \int_{S^n} 1[\mathbf{x} \in F] f(\mathbf{x}) \mathrm{d}x_1 \ldots \mathrm{d}x_n \quad (\mathbf{x} = \{x_1, \ldots, x_n\})$$

Example: Strauss process

For a point configuration $\mathbf{x}$ on a bounded region S, let $n(\mathbf{x})$ and $s(\mathbf{x})$ denote the number of points and number of (unordered) pairs of R-close points ($R \geq 0$).

A *Strauss process* $\mathbf{X}$ on S has density

$$f(\mathbf{x}) = \frac{1}{c} \exp(\beta n(\mathbf{x}) + \psi s(\mathbf{x}))$$

with respect to a unit rate Poisson process $\mathbf{Y}$ on S and

$$c = \mathbb{E} \exp(\beta n(\mathbf{Y}) + \psi s(\mathbf{Y})) \tag{4}$$

is the normalizing constant (unknown).

Note: only well-defined ($c < \infty$) if $\psi \leq 0$.

Intensity and conditional intensity

Suppose $\mathbf{X}$ has *hereditary* density f with respect to Y: $f(\mathbf{x}) > 0 \Rightarrow f(\mathbf{y}) > 0, \mathbf{y} \subset \mathbf{x}$.

Intensity function $\rho(u) = \mathbb{E} f(\mathbf{Y} \cup \{u\})$ usually unknown (except for Poisson and Cox/Cluster).

Instead consider *conditional intensity*

$$\lambda(u, \mathbf{x}) = \frac{f(\mathbf{x} \cup \{u\})}{f(\mathbf{x})}$$

(does not depend on normalizing constant !)

Note

$$\rho(u) = \mathbb{E} f(\mathbf{Y} \cup \{u\}) = \mathbb{E}\big[\lambda(u, \mathbf{Y}) f(\mathbf{Y})\big] = \mathbb{E}\lambda(u, \mathbf{X})$$

and

$\rho(u)\mathrm{d}A \approx P(\mathbf{X} \text{ has a point in } A) = \mathbb{E}P(\mathbf{X} \text{ has a point in } A | \mathbf{X} \setminus A), u \in A$

Hence, $\lambda(u, \mathbf{X})\mathrm{d}A$ probability that $\mathbf{X}$ has point in very small region A given $\mathbf{X}$ outside A.

Density and conditional intensity - factorization

One-to-one correspondence between density and conditional intensity (up to normalizing constant)

$$f(\{x_1,\ldots,x_n\}) \propto h(\{x_1,\ldots,x_n\}) = \prod_{i=1}^{n} \lambda(x_i,\{x_1,\ldots,x_{i-1})$$

Normalizing constant:

$$f(\mathbf{x}) = \frac{1}{c}h(\mathbf{x}) \quad c = \mathbb{E}h(\mathbf{Z})$$

Typically c is intractable so likelihood estimation is not straightforward.

Options: pseudo-likelihood (later in this section) or Monte Carlo approximation of c.

Markov point processes

Def: suppose that f hereditary and $\lambda(u, \mathbf{x})$ only depends on $\mathbf{x}$ through $\mathbf{x} \cap b(u, R)$ for some $R > 0$ (*local Markov property*). Then f is *Markov* with respect to the R-close neighbourhood relation.

Thm (Hammersley-Clifford) The following are equivalent.

1. f is Markov.
2. $$f(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \phi(\mathbf{y})$$

   where $\phi(\mathbf{y}) = 1$ whenever $\|u - v\| \geq R$ for some $u, v \in \mathbf{y}$.

Pairwise interaction process: $\phi(\mathbf{y}) = 1$ whenever $n(\mathbf{y}) > 2$.

NB: in H-C, R-close neighbourhood relation can be replaced by an arbitrary symmetric relation between pairs of points.

Modelling the conditional intensity function

Suppose we specify a model for the conditional intensity. Two questions:

1. does there exist a density f with the specified conditional intensity ?
2. is f well-defined (integrable) ?

Solution:

1. find f by identifying interaction potentials (Hammersley-Clifford) or guess f.

2. sufficient condition (local stability): $\lambda(u, \mathbf{x}) \leq K$

NB some Markov point processes have interactions of any order in which case H-C theorem is less useful (e.g. area-interaction process).

Some examples

Strauss (pairwise interaction):

$$\lambda(u,\mathbf{x}) = \exp\left(\beta+\psi\sum_{v\in\mathbf{x}} 1[\|u-v\|\le R]\right), \quad f(\mathbf{x}) = \frac{1}{c}\exp\left(\beta n(\mathbf{x})+\psi s(\mathbf{x})\right)$$

Overlap process (pairwise interaction marked point process):

$$\lambda((u,m),\mathbf{x}) = \frac{1}{c}\exp\left(\beta+\psi\sum_{(u',m')\in\mathbf{x}} |b(u,m)\cap b(u',m')|\right) \quad (\psi\le 0)$$

where $\mathbf{x} = \{(u_1,m_1),\ldots,(u_n,m_n)\}$ and $(u_i,m_i)\in\mathbb{R}^2\times[a,b]$.

Area-interaction process:

$$f(\mathbf{x}) = \frac{1}{c}\exp\left(\beta n(\mathbf{x})+\psi V(\mathbf{x})\right), \quad \lambda(u,\mathbf{x}) = \exp\left(\beta+\psi(V(\{u\}\cup\mathbf{x})-V(\mathbf{x})\right)$$

$V(\mathbf{x}) = |\cup_{u\in\mathbf{x}} b(u,R/2)|$ is area of union of balls $b(u,R/2)$, $u\in\mathbf{x}$.

NB: $\phi(\cdot)$ complicated for area-interaction process.

The Georgii-Nguyen-Zessin formula ('Law of total probability')

$$\mathbb{E} \sum_{u \in \mathbf{X}} k(u, \mathbf{X} \setminus \{u\}) = \int_S \mathbb{E}[\lambda(u, \mathbf{X}) k(u, \mathbf{X})] \,\mathrm{d}u = \int_S \mathbb{E}^![k(u, \mathbf{X}) \mid u] \rho(u) \,\mathrm{d}u$$

$\mathbb{E}^![\cdot \mid u]$: expectation with respect to the conditional distribution of $\mathbf{X} \setminus \{u\}$ given $u \in \mathbf{X}$ (*reduced Palm distribution*)

Density of reduced Palm distribution:

$$f(\mathbf{x} \mid u) = f(\mathbf{x} \cup \{u\}) / \rho(u)$$

NB: GNZ formula holds in general setting for point process on $\mathbb{R}^d$.

Statistical inference based on pseudo-likelihood

$\mathbf{x}$ observed within bounded S. Parametric model $\lambda_\theta(u, \mathbf{x})$.

Let $N_i = 1[\mathbf{x} \cap C_i \neq \emptyset]$ where C_i disjoint partitioning of $S = \cup_i C_i$.

$P(N_i = 1 \,|\, \mathbf{X} \setminus C_i) \approx \lambda_\theta(u_i, \mathbf{X} \setminus C_i)\mathrm{d}C_i$ where $u_i \in C_i$.

Hence composite likelihood based on the N_i:

$$\prod_{i=1}^{n} (\lambda_\theta(u_i, \mathbf{x} \setminus C_i)\mathrm{d}C_i)^{N_i}(1 - \lambda_\theta(u_i, \mathbf{x} \setminus C_i)\mathrm{d}C_i)^{1-N_i} \equiv$$

$$\prod_{i=1}^{n} \lambda_\theta(u_i, \mathbf{x} \setminus C_i)^{N_i}(1 - \lambda_\theta(u_i, \mathbf{x} \setminus C_i)\mathrm{d}C_i)^{1-N_i}$$

which tends to *pseudo-likelihood* function

$$\prod_{u \in \mathbf{x}} \lambda_\theta(u, \mathbf{x} \setminus \{u\}) \exp\big(- \int_S \lambda_\theta(u, \mathbf{x})\mathrm{d}u \big)$$

Score of pseudo-likelihood: unbiased estimating function by GNZ.

Pseudo-likelihood estimates asymptotically normal but asymptotic variance is not straightforward.

Integral approximated by numerical quadrature or Monte Carlo

Flexible implementation for log linear conditional intensity (fixed R) in `spatstat`

Estimation of interaction range R: profile likelihood (?)

Monte Carlo approximation

Let $\mathbf{D}$ 'quadrature/dummy' point process of intensity $\rho(\cdot)$ and independent of $\mathbf{X}$. $\mathbf{X} \cup \mathbf{D}$ has conditional intensity $\lambda(u, \mathbf{X}) + \rho(u)$

By GNZ

$$\mathbb{E} \int_W \lambda'(u, \mathbf{X}) \mathrm{d}u = \mathbb{E} \sum_{u \in \mathbf{X} \cup \mathbf{D}} \frac{\lambda'(u, \mathbf{X} \setminus \{u\})}{\lambda(u, \mathbf{X} \setminus \{u\}) + \rho(u)}$$

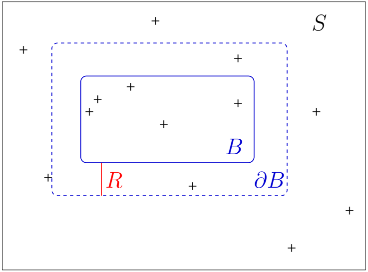
Idea: replace integral in pseudo-likelihood with unbiased estimates using $\mathbf{D}$.

Resulting estimating function formally equivalent to logistic regression

The spatial Markov property and edge correction

Let $B \subset S$ and assume $\mathbf{X}$ Markov with interaction radius R.

Define: ∂B points in $S \setminus B$ of distance less than R



Factorization (Hammersley-Clifford):

$$f(\mathbf{x}) = \prod_{\substack{\mathbf{y} \subseteq \mathbf{x} \cap (B \cup \partial B): \\ \mathbf{y} \cap B \neq \emptyset}} \phi(\mathbf{y}) \prod_{\mathbf{y} \subseteq \mathbf{x} \setminus B} \phi(\mathbf{y})$$

Hence, conditional density of $\mathbf{X} \cap B$ given $\mathbf{X} \setminus B$

$$f_B(\mathbf{z}|\mathbf{y}) \propto f(\mathbf{z} \cup \mathbf{y})$$

depends on $\mathbf{y}$ only through $\partial B \cap \mathbf{y}$.

Edge correction using the border method

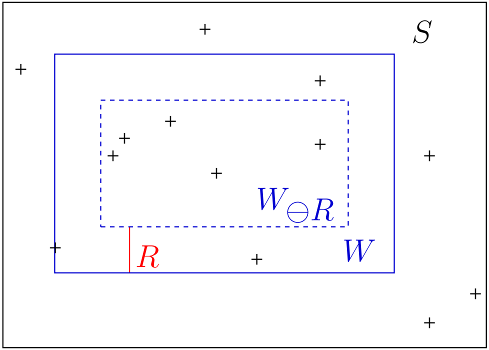
Suppose we observe $\mathbf{x}$ realization of $\mathbf{X} \cap W$ where $W \subset S$.

Problem: density (likelihood) $f_W(\mathbf{x}) = \mathbb{E} f(\mathbf{x} \cup Y_{S \setminus W})$ unknown.

Border method: base inference on

$$f_{W_{\ominus R}}(\mathbf{x} \cap W_{\ominus R} | \mathbf{x} \cap (W \setminus W_{\ominus R}))$$

i.e. conditional density of $\mathbf{X} \cap W_{\ominus R}$ given $\mathbf{X}$ outside $W_{\ominus R}$.



Exercises

1. Suppose that S contains a disc of radius $\epsilon \leq R/2$. Show that (4) is not finite, and hence the Strauss process not well-defined, when ψ is positive.

   (Hint: $\sum_{n=0}^{\infty} \frac{(\pi\epsilon^2)^n}{n!} \exp(n\beta + \psi n(n-1)/2) = \infty$ if $\psi > 0$.)
2. Show that local stability for a spatial point process density ensures integrability. Verify that the area-interaction process is locally stable.
3. what is the unnormalized density of a Strauss (β, ψ) with respect to a Poisson process of intensity $\exp(\beta)$?
4. Starting with the conditional intensity for a Strauss process, identify the potential function ϕ
5. (if time) Verify the Georgii-Nguyen-Zessin formula for a finite point process.

   (Hint: consider first the case of a finite Poisson-process $\mathbf{Y}$ in which case the identity is known as the Slivnyak-Mecke theorem, next apply $\mathbb{E}g(\mathbf{X}) = \mathbb{E}\big[g(\mathbf{Y})f(\mathbf{Y})\big]$.)

Solution: second order product density for Poisson

$$
\begin{aligned}
&\mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} 1[u \in A, v \in B] \\
=&\sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu(A \cup B)}}{n!} \int_{(A \cup B)^n} \sum_{u,v \in \{x_1,\ldots,x_n\}}^{\neq} 1[u \in A, v \in B] \prod_{i=1}^{n} \rho(x_i) \mathrm{d}x_1 \ldots \mathrm{d}x_n \\
=&\sum_{n=2}^{\infty} \frac{\mathrm{e}^{-\mu(A \cup B)}}{n!} 2 \binom{n}{2} \int_{(A \cup B)^2} \int_{(A \cup B)^{n-2}} 1[x_1 \in A, x_2 \in B] \prod_{i=1}^{n} \rho(x_i) \mathrm{d}x_1 \ldots \mathrm{d}x \\
=&\sum_{n=2}^{\infty} \frac{\mathrm{e}^{-\mu(A \cup B)}}{(n-2)!} \mu(A)\mu(B)\mu(A \cup B)^{n-2} \\
=&\mu(A)\mu(B) = \int_{A \times B} \rho(u)\rho(v) \mathrm{d}u \mathrm{d}v
\end{aligned}
$$

Solution: invariance of g (and K) under thinning

Since $\mathbf{X}_{\text{thin}} = \{u \in \mathbf{X} : R(u) \leq \pi(u)\}$,

$$
\begin{aligned}
&\mathbb{E} \sum_{u,v \in \mathbf{X}_{\text{thin}}}^{\neq} 1[u \in A, v \in B] \\
=&\mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} 1[R(u) \leq \pi(u), R(v) \leq \pi(v), u \in A, v \in B] \\
=&\mathbb{E}\,\mathbb{E}\Big[\sum_{u,v \in \mathbf{X}}^{\neq} 1[R(u) \leq \pi(u), R(v) \leq \pi(v), u \in A, v \in B] \,\Big|\, \mathbf{X}\Big] \\
=&\mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} \pi(u)\pi(v) 1[u \in A, v \in B] \\
=&\int_A \int_B \pi(u)\pi(v)\rho^{(2)}(u,v)\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}
$$

1. Intro to point processes and moment measures

2. The Poisson process

3. Cox and cluster processes

4. Estimating functions

5. The conditional intensity and Markov point processes

5. References

References

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(see the monograph M & W '03, and the two review papers, M & W '07, '16, for further references)